Mental Constructions Formed in the Conceptual Understanding of the Chain Rule

Zingiswa Mybert Monica Jojo

jojozmm@unisa.ac.za

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Abstract

This paper reports on how first year engineering University of Technology students, in South Africa conceptualised mathematical learning in the context of calculus with specific reference to the chain rule. Mental constructions made are analysed using reflective abstractions at the heart of which is APOS (Actions-Process-Object-Schema). This was a qualitative study where mental constructions made within twelve groups of 76 first year engineering students were analysed using a proposed initial genetic decomposition. The Triad was later used to analyse the constructions made in learning and understanding the chain rule. Results indicated that students could use and understand the chain rule even though they did not understand the composition of functions. This helped to inform pedagogy to be used in order to enable the students to understand the concept of the chain rule and improve their understanding in calculus.

1. Introduction

The key mechanism for an individual to obtain new mathematical meaning is for him/her to construct mental representations of direct experiences relevant to that concept. A structured set of mental constructs which might describe how the concept can develop in the mind of an individual is called the **genetic decomposition** of that particular concept. Dubinsky (1991) proposed that the mental constructions that the learner might make include Actions, Processes, Objects and Schemas (APOS). The genetic decomposition of the concept of the chain rule suggested below guided the researcher's teaching instruction in class and the construction of the interview and discussion tasks. APOS ascertains that to understand a mathematical concept begins with manipulating previously constructed mental or physical objects in the learner's mind to form **actions**; actions would then be interiorised to form **processes** which are then encapsulated to form **objects** (Dubinsky, 1991). These objects could be de-encapsulated back to the processes from which they are formed, which would be finally organized in **schemas**. Understanding the chain rule was explored in relation to the schema relevant to it. For an elaboration of these concepts refer to Maharaj (2010, p43).

The chain rule states that, if a function *g*, is differentiable at *a* and *f* is differentiable at *g(a)*, then $h = fog$ is a differentiable function at *a,* so that $h^{'}(a) = (fog)'(a) = f^{'}(g(a)).g^{'}(a)$. Hassani (1998) examined students' understanding on graphical, numerical (tabular) and algebraic/ symbolic presentations of composition of functions and the chain rule. His study revealed that first-year undergraduate calculus have a very meagre understanding of the composition of functions and their ability to explain or apply the chain rule is significantly related to their algebraic manipulative skills and their general knowledge of function concepts and function composition. Also Swanson (2006) asserted that the complexity of the chain rule deserves exploration because students struggle to understand it and because of its importance in the calculus curriculum. Despite the importance of the chain rule in calculus and its difficulty for students, the chain rule has been scarcely studied in mathematics educational research (Clark et al, 1997; Gordon, 2005; Uygur & Ozdas, 2007; Webster, 1978). These student difficulties include the inability to apply the chain rule to functions and also with composing and decomposing functions (Clark et al, 1997; Cottrill, 1999; Hassani, 1998).

The researcher's observations on students' performance in calculus using the chain rule has also revealed a difficulty in understanding and applying the concept as compared to other sections (exponential and logarithmic functions, trigonometry and complex numbers) in first year engineering mathematics. This led to the researcher's interest of how learners conceptualized the chain rule and how they could learn this concept effectively. The article therefore reports on how students can learn and apply the chain rule and thus inform the teaching of the concept.

Students were provided with activities in class that were designed to induce them to make the suitable mental constructions as suggested by the initial genetic decomposition. The tasks used in this study helped students gain experience in constructing actions corresponding to the chain rule.

Lastly, the students were then provided with complicated activities where they needed to organize a variety of previously constructed objects, like functions and derivatives of composition of functions, into a schema that could be applied to chain rule problem situations. More specifically the researcher examined students' attempts to answer the tasks given in class, their tests, and exercises with regard to their understanding of functions, composition of functions and the chain rule.

In the studies done on APOS using computers by (Cotrill et al, 1996), it was shown that the students were in a better position to make mental constructions using computers when finding graphically the limits of certain functions. The researcher did not use computers and was not sure about certain mental constructions not being constructed. Written responses were examined for errors made and interviews were conducted with selected candidates in relation to interesting responses displayed in the questionnaires. It is hoped that this engagement would fill in doubtful gaps of such a study.

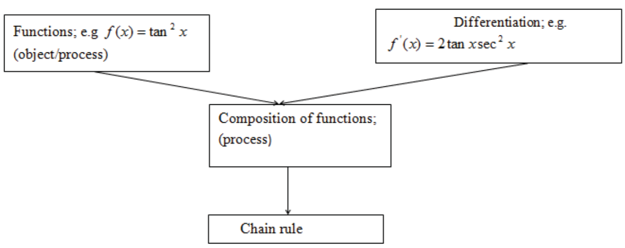
2. Theoretical Analysis

Piaget's work concentrated on the development of mathematics knowledge with children in early ages, and rarely going beyond adolescence. Also, pedagogical strategies are almost absent from the totality of Piaget's work (Asiala et al, 2004). The researcher felt that the emphasis on exploring the use of chain rule with trigonometric functions and verbal representations of calculus concepts can be fostered through reflective abstractions following Dubinsky's (1991b) model of conceptual understanding. He believes that the concept of reflective abstraction that was introduced by Piaget (Berth & Piaget, 1966) can be a powerful tool to describe the development of advanced mathematical thinking and that it could be used to analyse any mathematical knowledge applicable to higher education. This study has therefore adopted the APOS approach (Dubinky, 1991a), based on intuitive appeal as there has been little empirical research done documenting the use of it on students' conceptual understanding of the chain rule in the African continent. APOS has been used in research focusing on understanding of various mathematical concepts, (Pascual, 2004; Sfard, 1991; Tall, 1994; Dubinsky, 1991a; De Vries, 2001; Gray & Tall, 2002; Clark et al, 1997).

This study was conducted according to a specific framework for research and curriculum development in advanced mathematics education, which guided the systematic enquiry of how students acquire mathematical knowledge and what instructional interventions contribute to student learning. The framework consists of three components: **theoretical analysis, instructional treatment,** and **observations and assessment** of student learning as proposed by Asiala et al (2004).

It is in the theoretical analysis of the chain rule where the researcher presented an initial genetic decomposition (IGD) describing specific mental constructions which a learner should make in order to develop his/her conceptual understanding of the chain rule. These are modelled in the following diagram (Jojo, 2012).

Figure 1: Initial genetic decomposition of the chain rule



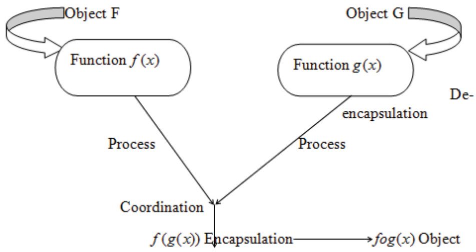
The IGD for understanding the chain rule proposed that understanding could begin with: (i) students understanding the composition of two or more functions, (ii) transformation of the functions to one composite function, (iii) understanding the derivative concept of the composite functions, and then (iv) coordination of the two processes to obtain this derivative which would then be encapsulated to using the chain rule for differentiation.

Construction of knowledge in this study was analysed through reflective abstraction at the heart of which is APOS (Dubinsky, 1991b) which then incorporates Piaget's **Triad** mechanism. Reflective abstraction has two components: (a) a projection of existing knowledge onto a higher plane of thought and (b) the reorganization of existing knowledge structures (Dubinsky, 1991a). Reflective abstraction is therefore a process of construction of knowledge, and Dubinsky outlines five kinds of construction in reflective abstraction: *Interiorisation*, *Co-ordination*, *Encapsulation*, *Generalisation* and *Reversal*. For full explanation of these concepts in chain rule concept knowledge construction, see Jojo , Brijlall and Maharaj, (2013).

The Triad mechanism occurring in three stages explained other constructions in the mind implicating mental representations and transformations in the analysis of schema formations. These stages are: The **Intra** stage focuses on 'a single entity', followed by **Inter-** which is 'study of transformations between objects' and **Trans-** noted as 'schema development connecting actions, processes and objects' Jojo et al (2013). Tall (1996) argues that there is no transformation that takes place when a child compares the sizes of objects. Zingiswa, Brijlall and Maharaj, (2005) assert that the idea of transformation is a tool the teacher can use, a tool that does not dehumanise learning because it corresponds to one aspect of the voluntary activities of the students' mind. Dubinsky (1991) believes that an individual at the Trans- stage for the function concept could construct various systems of transformations such as rings of functions, together with the operations included in such mathematical structures.

Jojo (2011) used the flow diagram (see Figure 2) to explain the activities involved in construction of the chain rule concept and to illustrate APOS extended.

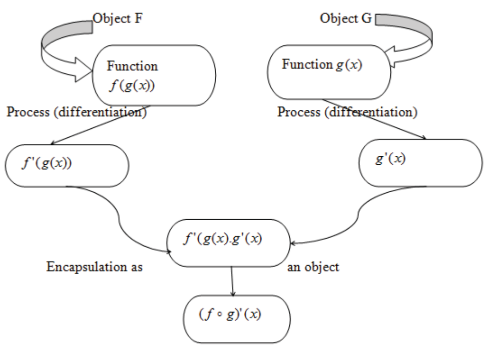
Figure 2: Illustration of the composition of functions



The overview of the model is that one begins with two functions F and G and transforms them into a single function, F o G. The transformation begins by de-encapsulation of F and G back to the process F(*x*) and G(*x*) from which it came. The two processes are then coordinated to obtain the process *x* on F(G(*x*)), which is then encapsulated to the object F o G (*x*).

The tricky part is in identifying the function $g(x)$ as a single entity in $f(x)$. The *x* in $f(x)$ is not *x* but another function in *x*. Thus the derivative $g'(x)$ has to be evaluated first and then multiplied by the derivative, $f'(g(x))$ to get $f'(g(x)).g'(x)$. To avoid the confusion of the two functions in *x*, one may represent the function *g(x)* as *u*. This will then enable us to find separately the derivatives $g'(x)$ and $f'(u)$. Result would then be found when we multiply $g'(x) \times f'(u)$. This would safe guard against two errors which the students usually make, namely: (1) Finding $f'(x) \times g'(x)$, for example the derivative, $\cos(3x^2 + 4x - 5) \neq -\sin x.(6x + 4)$ but is $-\sin(3x^2 + 4x - 5).(6x + 4)$. (2) Finding $f'(g'(x))$ where one derivative is plugged onto the other one. For example the derivative of $\tan(3x^4) \neq \sec^2 12x^3$but is $\sec^2(3x^4).12x^3$. Clear connections to the chain rule are made as the student collects all the objects formed as derivatives, multiplies the results to find the derivative of the composite function. The correct actions in total, applied and reversed with different equations, indicates acquisition of the constructed **schema** of the chain rule by the student modelled in figure 3.

Figure 2: A model of the chain rule



Students were provided with activities in class that were designed to induce them to make the suitable mental constructions suggested in an initial genetic decomposition.

The students interiorised their actions by discussing their actions with others collaboratively. They had to also write verbal descriptions of their actions using their own words. Mental constructions that a learner makes include actions, processes, objects and schema of the chain rule. Within APOS context, **actions** made externally in identifying a composition of functions should first be interiorised and reflected upon identifying functions within functions. A student should then **process** and transform these functions such that they are encapsulated in totality to form **objects** through differentiation.

3. Literature Review

Duffin and Simpson (2000) have identified and named three components of understanding as (1) the building, (2) the having, and (3) the enacting. They defined 'building understanding' as the formation of connections between internal mental structures. 'Having understanding' is said to be the state of these connections at any particular time and 'enacting understanding' as the use of the connections available at a particular moment to solve a problem or construct a response to a question. Thus this is the type of understanding that may be visible from students' work when responding to mathematical tasks. Duffin and Simpson also talked about the breadth and depth of understanding. They described the breadth of understanding to be determined by the number of different possible starting points that the learner may have in solving a problem. The depth may be evidenced by the way the learners could unpack each stage of their solution in more detail by referring to more concepts.

Imagine a situation where one is given a function say, $f(x) = (x^3 + 5x)^2$ to differentiate. A learner may identify this function as being represented structurally as $(x^3 + 5x)(x^3 + 5x)$, and may expand the expression before differentiating. This learner might then use the product rule to differentiate the resulting expression. Another learner who recognizes the function in the form of $f(g(x)) = (x^3 + 5x)^2$ where $f(x) = x^2$ and $g(x) = x^3 + 5x$, he/she may the use the chain rule to differentiate $f(x)$. A learner who sees this function to be represented structurally in one form only lacks breadth of understanding. Such a learner may even deny that the given equivalent forms of the function to be differentiated represent the same function. The depth of understanding in this case could be determined by the learner's ability to state at each stage what is happening in mathematical terms. For example, a learner could indicate the stages at which the power rule, the product rule or the chain rule have been applied, that is, alongside the work shown in performing the mathematical task. This demonstrates a deeper understanding of solving the task than in a case where the structures will be manipulated by applying a rule with no explanations at all. A learner who instrumentally carries out manipulations is likely to be unaware of the mistakes he or she has committed.

On the contrary the understanding of a mathematical concept is explained in this study with the help and adoption of APOS. It can be also agreed (Dubinsky & McDonald, 2001) that mathematical ideas begin with human activity and then proceed to be abstract concepts. It is therefore important for us to understand how the construction of concepts in the mind, lead to abstraction of mathematical knowledge. This interpretation of the relevant knowledge construction processes is essential since it points to the contributions we get from APOS analysis. These include (1) understanding the importance of human thought, (mental constructions relating to the concept), and (2) pointing to effective pedagogy for a particular concept.

The aim of this paper is *to report on students' construction of an underlying structure of the chain rule.* This focus was accomplished by:

1) A discussion of the types of structures constructed by students when learning the chain rule with the view to clarifying their understanding of: (i) the composition of function and (ii) the derivative.
2) Finding out how the lack or availability of these structures hamper or assist students' understanding of the chain rule.
3) Determining the students' actual engagement with tasks and how these tasks link with the expected outcomes highlighted in the initial genetic decomposition.

4. Participants, Instructional design Methodology

The study was qualitative in nature targeting 176 first year electrical engineering University of Technology students in KwaZulu Natal in 2011. Data was collected in the form of **A**ctivities, **C**lassroom discussions and **E**xercises done out of class, (ACE). These were conducted in the form of tutorial tests on tasks pre-supposed to test students' understanding of (1) composition and decomposition of functions, (2) differentiation, and (3) use and applications of chain rule in differentiated of loaded trigonometric functions. These were administered during tutorial sessions held each Friday afternoons in 1hour slots as a pedagogical approach based on APOS. These tutorial lessons were held to reinforce learning and teaching done during the instruction of the calculus concepts in normal lectures so the sampling was convenient. Students in these sessions were divided into 6 groups of +/- 30 groups each supervised by different tutors simultaneously in different classroom venues. It was not important whether the responses were correct of wrong, of more significance, was the procedure students used. Interviews were then conducted with five participants chosen on the basis of their responses to the written instruments.

This section uses tables, written extracts, and interviews to describe the comprehension of students with regard to the use and application of the chain rule concept.

5. Analysis and Discussion

The worksheets were analyzed for meaning which is one of the mechanisms necessary for understanding a concept. These included detecting (1) the connections made by students to other concepts, (2) calculations made using the chain rule, (3) the chain rule technique used, and (4) mental images on which the chain rule was based, (5) composition and decomposition of functions.

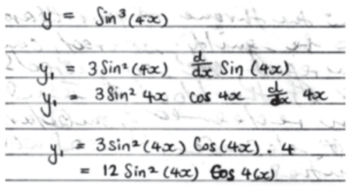
5.1 Discussion on Task 1

Differentiate: $y = \sin^3(4x)$

Table 1 summarizes the question analysis of task 1 using the responses presented by the groups in this task.

	Incorrect responses	Partially correct	Completely correct	Chain rule preference	Connection to other concepts
Number of groups	3	4	2	1	8
% groups	0	16,7	83,3	100	66,7

Table 2: Analysis of task 2



Interviews with the student whose work has been displayed on the extract revealed his deeper understanding of the technique he used in differentiating.

Researcher: I notice that you used Leibniz technique in finding the derivative. Why?

Lucy: *I am always sure ukuthi* (that) *if I use it, I will always know which function I have already differentiated and then follow with the other one, and so on.*

Researcher: How else could you have done it?

Lucy: *Oh well, this is an easy one, so I would start with the power, pull it down, ten follow with sin, then 4x and multiply everything*

Researcher: Would you get the same answer?

Lucy*: Oh yes, why not, I can show you….*

The one group that presented an incorrect response, left out the square sign after differentiating with respect to the power of the *sin* function. Their actions were not interiorized with regards to the derivative concept and this had an impact on applying the chain rule in the given task. Their mental images could not be related to the string of symbols forming the expression, since they could not interpret both the symbols and or manipulations. Since calculations reflect the active part of mental constructions, the differentiation rules for these students were not perceived as entities on which actions could be made. Dubinsky (2010) asserts that in such cases the difficulty does not depend on the nature of the formal expressions, but rather in the loss of the connections between the expressions and the situation instructions.

5.2 Discussion on Task 2

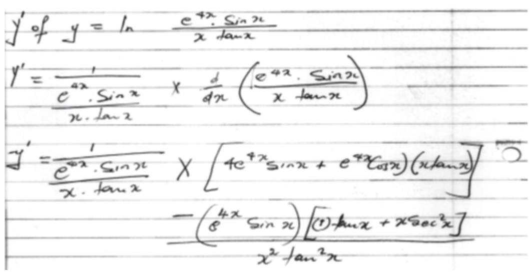
Differentiate: $$y = \ln \frac{e^{4x} \sin x}{x \sin x}$$

Table 4 summarizes the question analysis of task 4 using the responses presented by the groups in this task.

	Incorrect responses	Partially correct	Completely correct	Chain rule preference	Connection to other concepts
Number of groups	3	4	2	1	8
% groups	25	33,3	16,7	100	66,7

Table 2: Analysis of task 2

Generally, one of two strategies was employed by students. The first form technique called for a specific connection between application of natural logarithms and differentiation. 16,7% of the groups displayed a coherent collection of the logarithmic rules and differentiation. Those groups were operating in the Trans- stage since they reflected on the explicit structure of the chain rule and were also able to operate on the mental constructions which made up their collection. Those students presented responses showing internal processes for manipulating logarithmic objects. Their schema enabled them to understand, organize, deal with and make sense out of application of the product rule, quotient, logarithmic rules and the chain rule. The other three groups could not apply logarithmic rules correctly and as such could not process the differentiation of the given task. The interpretation of logarithms was then incorrect since a bracket was left out in step three of the response. Thus the function differentiated was not the originally given one. Even in their process of differentiation some brackets were still left out when they should have been there. Those students did not recognize the relationships between application of natural logarithms and algebraic manipulations resulting in multiplications when they were due and subtractions where appropriate. They perceived differentiation as a separate entities and even the rules applied were not remembered correctly. These were operating in the Intra- stage of the Triad. Extract 2 below illustrates a response where the students differentiated with respect to the natural logarithm first and always employed the Leibniz form technique until all functions were differentiated.



Extract 2: Using the Leibniz technique of chain rule in differentiation

5.3 Discussion on Task 3

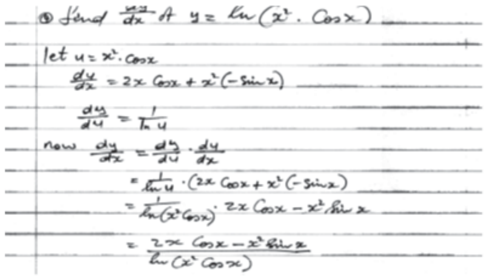
Differentiate: $$y = \ln(x^2 \cos x)$$

Table 3 summarizes the question analysis of task 3 using the responses presented by the groups in this task.

	Incorrect responses	Partially correct	Completely correct	Chain rule preference	Connection to other concepts
Number of groups	1	2	9	12	7
% groups	8,3	16,7	75	100	58,3

Table 3: Analysis of task 3

Although most of the groups (9 out of 12) differentiated correctly, brackets and format of the response were neglected.



In this extract, differentiation was done using the Leibniz substitution form. The student here displayed interiorisation of actions in both algebraic manipulations, use of brackets and negative signs where necessary. He/she therefore has an object understanding of the chain rule and can apply it.
Some students misrepresented the derivative of $\cos x$ *as* $\sin x,$ they left out the negative sign. Those students were just differentiating as an action not taking care and without constructing a meaning into it. For them it's just using rules and knowing that the derivative of this function is just that. There were no processes coordinated. Those students were operating in the action stage since they saw the given function as a formula and the errors of signs left out where they should be, meant that those actions were not interiorized to processes. According to the Triad, they were operating in the Intra- stage since the students had a collection of rules for finding derivatives of functions in various situations, but had no recognition of the relationships between them.

5.4 Discussion on Task 4

Differentiate: $y = \cot^3(x^5 + e^{\sqrt{x^2+3}})$

Table 2 summarizes the question analysis of task 4 using the responses presented by the groups in this task.

	Incorrect responses	Partially correct response	Completely correct response	Chain rule preference in differentiation	Connection to other concepts
% groups	50	33,3	16,7	100	58,3

Table 2: Analysis of task 4

All the groups applied the chain rule to the first task incorrectly differentiating in a straight form from left to right. It was only 16,7% of the groups that presented completely correct responses. One of the groups who left out the bracket then went on to detach the derivative $5x^4$ of x^5 from the + sign. These mistakes were not detected by any of the other members of the same group. Those students struggled with the connection of previously learnt algebraic skills like use of brackets where appropriate and manipulation of algebraic terms in a function. The calculations presented after differentiating using the chain rule successfully were therefore not correct for 83,3% responses received. When one representative was interviewed and asked to state the chain rule, he wrote: 'I just differentiate from left to right and multiply.

The actions explained cannot guarantee the correct application of the chain rule especially when the task calls for use of logarithms. Such students are said to operate in the action stage of APOS with their actions on algebraic manipulations not interiorised and as such could not be applied with other concepts.

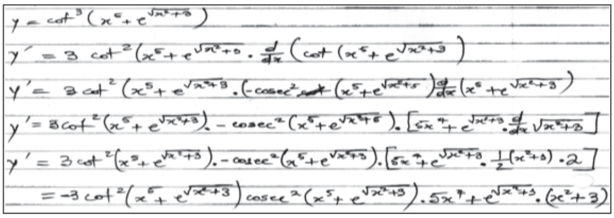
It was very interesting to mark one group's response to this task. When the researcher asked one member of the group to explain how they reached their response, he said, '*First we take care of the power, mhm……*

Researcher: yes, go on…

S1: *Ngiya differentiyeyitha ipower kuqala, bese wonke ama functions engingekawenzi, ngiloko ngibhala u* $\frac{d}{dx}$ *for* *everything that I have not differentiated njalo, ngize ngiqede. (I differentiate with respect to the power first, and always write* $\frac{d}{dx}$ *for every function that I haven't differentiated until I finish). Angithi yiyona ichain rule leyo, mangabe ngizwe kahle? (Is that not the chain rule?)*

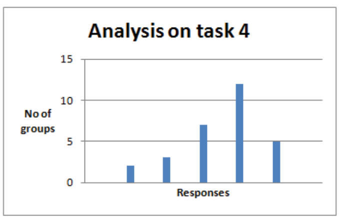
Researcher: Oh well I am impressed except that the last part, (pointing at 2)…

S1: *Sorry mhem, iphutha lami,(my mistake), supposed to be 2x. (He said that grabbing his pen quickly and rectifying the mistake, putting 2x instead of 2.*



Extract 3 Group response

In step two of his solution they left out the second bracket but recaptured it back in the following steps. This was a common error made by other students regarding change in operations after differentiation. Also they presented the derivative of $\sqrt{x^2+3}$ as $\frac{1}{2}(x^2+3)$ instead of $\frac{1}{2}(x^2+3)^{-\frac{1}{2}}$. The latter error led them to multiply $\frac{1}{2}\times 2$ and got (x^2+3). Their solution is a guided notation using a form technique of chain rule application and was used with caution initially. They demonstrated mental construction of the chain rule as an object. At this point in his (group representative) development he displayed the ability to reverse certain processes. He could trace back the steps of which functions were already and still to be differentiated. According to the Triad, they operated in the Inter- stage with regards to chain rule application since they displayed construction of the underlying structure of the chain rule as an object through reflection on relationships between various processes from previous stages. This was done by putting $\frac{d}{dx}$ before functions still to be differentiated. The two errors are not associated with chain rule applications but ascertain that they did not verify their response. They seemed excited and sure about chain rule application. The summary on means after coding the results, analysis for all groups is displayed in the following graph.



6. Conclusion

Students' understanding of the chain rule was not connected to their understanding of the composition of functions. They could not decompose the given functions but they used the chain rule correctly except for minor errors emanating from lack of underlying structures of basic algebra. A common error where students recorded the derivative of *cos x* correctly as – *sin x* but left out the brackets to end up with a different function from the one that was given for differentiation, was observed. Such students' actions of differentiation are detached from the basic algebraic operational signs. The multiplication sign left out indicates the absence of links between actions and procedures. Knowing the derivative of a particular function is not an indication of conceptual understanding since the relationships constructed internally were not connected to existing ideas. This understanding should also involve the knowledge and application of mathematical ideas and procedures related to basic arithmetic facts.

It was also noticed that most students in different groups were operating in the Intra- stage of the Triad. They had a collection of rules of differentiation with no recognition of relationships between them. Those students were helped by others who reflected on using the chain rule by applying the input by other students to group dynamics. The latter group had created an object of the chain rule. At the same time they applied actions on differentiation and as such the process of differentiating using the chain rule was encapsulated to form an object.

A possible modification to the proposed genetic decomposition was made. The student recognizes and applies the chain rule to specific situations using either the *straight, link or Leibniz* form techniques. This would then help the student to think of an interiorised process of differentiation in reverse and to construct a new process by reversing the existing one. Instruction on the conceptual understanding of the chain rule should incorporate all three different techniques. It would also be necessary to revisit instruction on basic manipulations of algebraic terms stressing the use of brackets where appropriate.

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