

## Use of the Chebyshev Polynomial Approximation in the Analysis of Milk Production and Average Weight at Birth Calves Depending on the Volume Food and the concentrate

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### Abstract

*In this study published the influence of two nutritional factors of milk production and the use of the production functions in the farming sector. The production, profit, and management are key elements for efficient economy. The main purpose of the project is to use contemporary methods for economic analyses of the use of resources (inputs), more specifically in small family farms. Nowadays in order to have a sustainable development of the agricultural farms, especially the livestock farms, it requires an economic optimization, as well as continuous analyses of economic and technical of influential factors. The main method used on the above mentioned study is the ones of Cobb-Douglas production function. This method analyses the impact of nutritional factors (the structure of the nutritive ration; wet, dead, concentrate) on bovine milk production. As illustrated in the article the economic theory combined with the deep mathematic logic are two main directions for the treatment of production functions. The study was conducted in "Agrotex" farm in Lushnje district. They were processed and analyzed data on the feeding phase and data on milk yield and weight of calves at birth for a period of 5 years. They were processed and analyzed data feeding phases mentioned above and milk yield and weight of calves at birth. These data were analyzed for a period of 5 years (period 2008-2012) with a number of heads of: 103, 111, 176, 118, 139 respectively in 2008, 2009, 20010, 20011, 2012. During the study is used Cobb-Douglas production function to analyze the impact of two components ration on milk production and approaches in the Chebishev's polynomials to solve systems of constraints. It appears that the most appropriate structure of the ration, to provide an average yield of 6478.3 kg milk / lactation (305 days), or an average of 21.3 kg milk / day, should be as follows: 17558.7 kg voluminous food and = 2265.7 kg concentrate. We prove that maximum profit and minimum cost to achieve food = 17558.7 kg = 2265.7 kg voluminous and concentrated. In this case the amount of milk produced by a cow will be 6478.3 kg while the average weight of calves at birth will be 43.8 kg. This study confirms that balanced nutrition is the primary factor in increasing economic efficiency of farms. As a result, the study demonstrates that the maximum profits, as well as the maximum income, are reached at the same point in the expansion path, where the cost is minimal.*

**Keywords:** optimal structure, milk production, nutritive factors, approaches in the Chebyshev's, optimal production, food ration

### 1. Introduction

Milk production mainly from cattle is different in different areas of the country. Production in lowland areas has been increased as a result of the increased number of heads of cattle, their production and improving of the food base.

The phased nutrition is a program which means the feeding of the flock in the time period based on the level of the milk production, the milk fat content, the amount of the consumed food and the live weight of the animal. Producers / farmers should develop the rations so that they can meet the needs of the animals in each of these stages for an optimum output, to minimize or to avoid metabolic disorders, to increase the length of the animal life and increase profit by batches.

### 2. Material And Method

The study was conducted in "Agrotex" farm in Lushnje district. They were processed and analyzed data on the feeding phase and data on milk yield and weight of calves at birth for a period of 5 years. They were processed and analyzed data feeding phases (1. Up to 150 days lactation, 2. Above 150 day lactation, 3 . the drying period) the milk yield and weight of calves at birth. These data were analyzed for a period of 5 years (period 2008-2012) with a number of heads of: 103, 111, 176, 118, 139 respectively in 2008, 2009, 20010, 20011, 2012. During the study is used Cobb-Douglas production function to analyze the impact of two components ration on milk production and approaches in the Chebishev's polynomials to solve systems of constraints.

### 3. Results And Discussion

- Were defined the forms of the functions of the milk production and the average weight of calves at birth.
- It was confirmed the suitability of the chosen models.
- We found the optimum combination of the inputs (ration structure) to maximize the profit and to minimize the cost.
- Were grouped, analyzed and processed the average values of the milk production, the average quantities of food during a twelve month period, according to three phases of handling a cow: (1. Up to 150 days lactation, 2. Above 150 day lactation, 3 . the drying period) and the average weight of calves at birth in the twelfth month for those cows that were born in the twelfth month.

In table 1 are the voluminous amount of food, the concentration, the average milk production and the average weight of calves at birth.

**Table 1:** The milk production and the average weight of calves at birth

Nr	The voluminous food	The concentration	The milk production	The average weight of calves at birth
1	11715	3771	6051.3	42.8
2	11775.3	3832.2	6215.1	43.3
3	11743.8	3801.6	6083.7	43.4
4	11974.2	3862.8	6083.7	43
5	11801.4	3801.6	6051.3	43.2
6	12147.9	4138.2	6215.1	44
7	12147.9	4015.8	6182.7	43.5
8	12175.8	4077	6248.4	43.5
9	12233.4	4015.8	6182.7	43
10	12378.3	4138.2	6248.4	43.5
11	12493.5	4260.6	6314.1	44
12	12291.9	4046.4	6280.8	43.9
13	12378.3	4260.6	6378.9	44.3
14	12493.5	4352.4	6412.2	44.6
15	12319.8	4107.6	6280.8	43.8
16	12407.1	4168.8	6346.5	45
17	12405.3	4230	6378.9	44.8

18	12378.3	4168.8	6346.5	44.2
19	11265.6	3465.2	5882.6	41.9
20	11139.9	3433.5	5788.5	41.2
21	10908.6	3341.7	5755.2	41
22	10474.8	3096.9	5525.7	39.2
23	10793.4	3249.9	5722.8	40.7
24	11369.4	3555.9	5985.6	42.6
25	10618.8	3158.1	5591.4	39.7
26	11571	3893.4	6018	42.8
27	10907.7	3341.7	5722.8	40.6
28	10822.2	3280.5	5722.8	40.5
29	10793.4	3188.7	5689.5	40.3
30	11341.5	3771	5952.3	42.3
31	11022.9	3402.9	5755.2	40.9
32	11081.4	3402.9	5788.5	40.6
33	10995	3372.3	5755.2	41
34	11399.1	3587.4	5919.9	42
35	11023.8	3341.7	5788.5	40.8
36	11542.2	3679.2	5985.6	42.1
37	10590	3158.1	5558.1	39.3
38	11429.7	3618	5952.3	42.4
39	10677.3	3219.3	5591.4	39.7
40	11513.4	3771	6051.3	43.2
41	12579.9	4260.6	6346.5	45.4
42	12666.3	4321.8	6378.9	45.2
43	12608.7	4291.2	6378.9	45.5
44	12724.8	4352.4	6510.3	46.1
45	12752.7	4383	6444.6	45.6
46	12696	4383	6412.2	45.7
47	12811.2	4413.6	6510.3	46.6
48	11601.6	3801.6	6083.7	43.3

The average prices for 1 kg of wet food, for 1 kg of dry food and for 1 kg of concentrate are 3.8 L, 4.8 L and 15.2 L. Production function was made in the form  $y = Ax_1^\alpha x_2^\beta x_3^\gamma$

It was used the linear regression method to determine  $\log A, \alpha, \beta$  and  $\gamma$ , through the econometric and computer software package SPSS. Resulted that the models are suitable. Given from these data were built the two production functions given below:

$$y_1 = 23.31274017 \cdot x_1^{0.424} \cdot x_2^{0.192}$$

$$y_2 = 0.3201390009 \cdot x_1^{0.331} \cdot x_2^{0.218}$$

where  $x_1, x_2, y_1$  and  $y_2$  are marked respectively the amount of voluminous feed, of the concentrate, the average milk production and the average weight of calves at birth. Was verified the hypothesis on the importance of the regression, and it was shown that at least one of the variables gives the information for the prognosis of the  $y$ , that is to say that the model is useful for predicting the value of  $y$ . Also, is confirmed the hypothesis of the importance of the parameters of the model.

#### 4. Minimize of the cost

We have the functions:  $y_1 = Ax_1^\alpha x_2^\beta, y_2 = Bx_1^a x_2^b$

Notice:  $y_1 = f(x_1, x_2), y_2 = g(x_1, x_2)$

Remark with  $r_1$  and  $r_2$  respectively the prices of the inputs  $x_1$  and  $x_2$ , and with  $p_1$  and  $p_2$  respectively the prices of the outputs.

We formed the Lagrange LC to have the minimum of the cost:

$$LC = r_1x_1 + r_2x_2 + \lambda_1[y_1 - f(x_1, x_2)] + \lambda_2[y_2 - g(x_1, x_2)]$$

and we expressed the necessary conditions for the minimum of the LC:

$$\frac{\partial LC}{\partial x_1} = r_1 - \lambda_1 f'_{x_1}(x_1, x_2) - \lambda_2 g'_{x_1}(x_1, x_2) = 0$$

$$\frac{\partial LC}{\partial x_2} = r_2 - \lambda_1 f'_{x_2}(x_1, x_2) - \lambda_2 g'_{x_2}(x_1, x_2) = 0$$

$$\frac{\partial LC}{\partial \lambda_1} = y_1 - f(x_1, x_2) = 0$$

$$\frac{\partial LC}{\partial \lambda_2} = y_2 - g(x_1, x_2) = 0$$

From the last two equations we expressed  $x_1$  and  $x_2$  depending on  $y_1$  and  $y_2$ .

$$\begin{aligned} y_1 &= Ax_1^\alpha x_2^\beta \Rightarrow \left(\frac{y_1}{A}\right)^{\frac{1}{\beta}} = x_1^{\frac{\alpha}{\beta}} x_2 \Rightarrow x_1^{\frac{\alpha b - a\beta}{b\beta}} = \left(\frac{y_1}{A}\right)^{\frac{1}{\beta}} \cdot \left(\frac{y_2}{B}\right)^{-\frac{1}{b}} \\ y_2 &= Bx_1^a x_2^b \Rightarrow \left(\frac{y_2}{B}\right)^{\frac{1}{b}} = x_1^{\frac{a}{b}} x_2 \end{aligned}$$

Are considered the sufficient conditions for the minimum of the LC with side of the determinant of the border Hessian and we have proved that the cost function has the minimum in the values  $x_1$  and  $x_2$  expressed in equations(\*)

$$x_1 = \left(\frac{y_1}{A}\right)^{\frac{b}{\alpha b - \beta a}} \cdot \left(\frac{y_2}{B}\right)^{-\frac{\beta}{\alpha b - \beta a}}$$

$$x_2 = \left(\frac{y_1}{A}\right)^{-\frac{a}{\alpha b - \beta a}} \cdot \left(\frac{y_2}{B}\right)^{\frac{\alpha}{\alpha b - \beta a}}$$

(\*)

$$\lambda_1 = \frac{1}{y_1(\alpha b - \beta a)} \left[ b r_1 \left(\frac{y_1}{A}\right)^{\frac{b}{\alpha b - \beta a}} \left(\frac{y_2}{B}\right)^{-\frac{\beta}{\alpha b - \beta a}} - a r_2 \left(\frac{y_1}{A}\right)^{-\frac{a}{\alpha b - \beta a}} \left(\frac{y_2}{B}\right)^{\frac{\alpha}{\alpha b - \beta a}} \right]$$

$$\lambda_2 = -\frac{1}{y_2(\alpha b - \beta a)} \left[ \beta r_1 \left(\frac{y_1}{A}\right)^{\frac{b}{\alpha b - \beta a}} \left(\frac{y_2}{B}\right)^{-\frac{\beta}{\alpha b - \beta a}} - \alpha r_2 \left(\frac{y_1}{A}\right)^{-\frac{a}{\alpha b - \beta a}} \left(\frac{y_2}{B}\right)^{\frac{\alpha}{\alpha b - \beta a}} \right]$$

$$|H| = \begin{vmatrix} 0 & 0 & \frac{\alpha y_1}{x_1} & \frac{\beta y_1}{x_2} \\ 0 & 0 & \frac{\alpha y_2}{x_1} & \frac{\beta y_2}{x_2} \\ \frac{\alpha y_1}{x_1} & \frac{\alpha y_2}{x_1} & L''_{x_1 x_1} & L''_{x_1 x_2} \\ \frac{\beta y_1}{x_2} & \frac{\beta y_2}{x_2} & L''_{x_2 x_1} & L''_{x_2 x_2} \end{vmatrix} = \left(\frac{y_1 y_2}{x_1 x_2}\right)^2 (\alpha b - a\beta)^2 > 0$$

### 5. The profit maximization

We have examined the function of profit to maximize the profit:

$$F = p_1 f(x_1, x_2) + p_2 g(x_1, x_2) - r_1 x_1 - r_2 x_2$$

Are considered necessary conditions for the maximum of the profit.

$$\frac{\partial F}{\partial x_1} = p_1 f'_{x_1}(x_1, x_2) + p_2 g'_{x_1}(x_1, x_2) - r_1 = 0$$

$$\frac{\partial F}{\partial x_2} = p_1 f'_{x_2}(x_1, x_2) + p_2 g'_{x_2}(x_1, x_2) - r_2 = 0$$

So we have to solve the system (\*\*):

$$\begin{cases} p_1 \alpha y_1 + p_2 a y_2 = r_1 x_1 \\ p_1 \beta y_1 + p_2 b y_2 = r_2 x_2 \\ y_1 = A x_1^\alpha x_2^\beta \\ y_2 = B x_1^a x_2^b \end{cases} \quad (**)$$

We have seen if completed the sufficient conditions for the maximum of the profit before we solve the system (\*\*).

For this we determined the sign of the determinant of Hessian and its main minors.

$$\frac{\partial^2 F}{\partial x_1^2} = \frac{\alpha(\alpha-1)p_1 y_1}{x_1^2} + \frac{a(a-1)p_2 y_2}{x_1^2}$$

$$\frac{\partial^2 F}{\partial x_2^2} = \frac{\beta(\beta-1)p_1 y_1}{x_2^2} + \frac{b(b-1)p_2 y_2}{x_2^2}$$

$$\frac{\partial^2 F}{\partial x_1 \partial x_2} = \frac{\alpha \beta p_1 y_1}{x_1 x_2} + \frac{a b p_2 y_2}{x_1 x_2} = \frac{\partial^2 F}{\partial x_2 \partial x_1}$$

Since  $\alpha = 0.424, \beta = 0.192, a = 0.331, b = 0.218$  appears that  $\frac{\partial^2 F}{\partial x_1^2} < 0$  and

$$|H_2| = \begin{vmatrix} \frac{\partial^2 F}{\partial x_1^2} & \frac{\partial^2 F}{\partial x_1 \partial x_2} \\ \frac{\partial^2 F}{\partial x_2 \partial x_1} & \frac{\partial^2 F}{\partial x_2^2} \end{vmatrix} > 0$$

So, the maximum of the profit is achieved for values  $x_1$  and  $x_2$  that are solutions of the system (\*\*). To (\*\*) can be computed the values  $f(x_1, x_2)$  and  $g(x_1, x_2)$ .

Then we passed to the resolution of the system (\*\*).

Technically it is difficult to gauge the exact solution of system (\*\*) so we find its approximate solution to a satisfactory approximation by the Chebishev polynomials, which has made it possible to identify areas where moves  $y_1$  and  $y_2$ . These areas have served as the options to find the exact solution of the system (\*\*) through MAPLE program.

Now let's implement the polynomial approximation of Chebishev hand to draw relevant conclusions. Polynomials of Chebishev are  $T_n(x) = \cos(n \cdot \arccos x)$ . So for example  $T_0(x) = 1, T_1(x) = x, T_2(x) = 2 \cdot x^2 - 1, T_3(x) = 4 \cdot x^3 - 3 \cdot x$ . In general for polynomial  $T_n(x)$  through the recurrence formula we have:  $T_n(x) = 2 \cdot x \cdot T_{n-1}(x) - T_{n-2}(x)$ . Indeed if the identity  $\cos(n\theta) + \cos((n-2)\theta) = 2 \cos \theta \cos((n-1)\theta)$  substitute  $\theta = \arccos x$  we have the above equations. On the segment  $[-1, 1]$  there are the points  $\xi_1, \xi_2, \dots, \xi_n$ , which are the zeros of the polynomial

$T_n(x)$  ie  $\xi_i = \cos\left(\frac{2i-1}{2n}\pi\right), i=1,2,\dots,n$ . Let it be given function  $f(x)$  defined on the segment  $[a, b]$ . Through of the transformation  $x = a + \frac{1}{2}(b-a)(\xi + 1)$  the faces  $\xi_1, \xi_2, \dots, \xi_n$ , on  $[-1,1]$  reflect to the points  $x_1, x_2, \dots, x_n$ , on  $[a, b]$ , where  $x_i = a + \frac{1}{2}(b-a)(\xi_i + 1)$  or  $x_i = \frac{1}{2}(b+a) + \frac{1}{2}(b-a)\xi_i$  (because this transformation reflect  $[-1,1]$  to  $[a,b]$ ). If the function  $f(x)$  has at any point of the segment  $[a, b]$  continuous derivatives of the n-th order and satisfying  $|f^{(n)}(x)| \leq M_n$  then the polynomial  $p_{n-1}(x)$  of degree n-1 that interpolon points  $(x_1, f(x_1)), (x_2, f(x_2)), \dots, (x_n, f(x_n))$ , satisfies the condition  $|f(c) - p_{n-1}(c)| \leq \frac{M_n}{2^{(2n-1)}n!}(b-a)^n$  where c is the point of  $[a, b]$ .

To solve the system (\*\*\*) by approximating  $y_1$  and  $y_2$  with the Chebishev polynomials  $\sum_{i=0}^n c_i T_i(x)$  where the coefficients determined by the equations,

$$c_0 = \frac{1}{n+1} \sum_{k=0}^n f(x_k) T_0(x_k) = \frac{1}{n+1} \sum_{k=0}^n f(x_k) \quad c_i = \frac{2}{n+1} \sum_{k=0}^n f(x_k) T_i(x_k) \quad \text{ku } i = 1, 2, \dots, n.$$

Starting from the outcome of the case: 3 inputs and 1 output as  $[a, b]$  to  $x_1$  we take  $[15250.15350]$ , while to  $x_2$  we take  $[1650.1750]$ . In our case we will have to

$$x_{1\ 0.424} = 59.4714988 T_0(u) + 0.08240510611 T_1(u) - 0.00003877896659 T_2(u) + 0.332874762 \cdot 10^{-7} T_3(u).$$

$$x_{2\ 0.192} = 4.170917675 T_0(v) + 0.02355793012 T_1(v) - 0.0001399909463 T_2(v) + 0.1240967387 \cdot 10^{-5} T_3(v).$$

$$x_{1\ 0.331} = 24.27351146 T_0(u) + 0.02625669614 T_1(u) - 0.00001435112187 T_2(u) + 0.1304579722 \cdot 10^{-7} T_3(u).$$

$$x_{2\ 0.218} = 5.060831165 T_0(v) + 0.03245494687 T_1(v) - 0.0001866545327 T_2(v) + 0.1630827199 \cdot 10^{-5} T_3(v).$$

Where  $u = \frac{1}{50}x_1 - 306$  dhe  $v = \frac{1}{50}x_2 - 34$ , and  $T_0(u) = 1, T_0(v) = 1$

$$T_1(u) = \frac{1}{50}x_1 - 306, T_2(u) = 2\left(\frac{1}{50}x_1 - 306\right)^2 - 1, T_3(u) = 4\left(\frac{1}{50}x_1 - 306\right)^3 - \frac{3}{50}x_1 + 918 \quad \text{dhe}$$

$$T_1(v) = \frac{1}{50}x_2 - 34, T_2(v) = 2\left(\frac{1}{50}x_2 - 34\right)^2 - 1, T_3(v) = 4\left(\frac{1}{50}x_2 - 34\right)^3 - \frac{3}{50}x_2 + 102$$

To replacing the approaches by the Chebishev polynomials in third and fourth equation of the system (\*\*\*) we will have to

$$10.6 y_1 + 115.85 y_2 = 4.2 x_1$$

$$4.8 y_1 + 76.3 y_2 = 15.2 x_2$$

$$y_1 = 23.31274017(59.4714988 T_0(u) + 0.08240510611 T_1(u) - 0.00003877896659 T_2(u) + 0.332874762 \cdot 10^{-7} T_3(u)) - (4.170917675 T_0(v) + 0.02355793012 T_1(v) - 0.0001399909463 T_2(v) + 0.1240967387 \cdot 10^{-5} T_3(v))$$

$$y_2 = 0.3201390009(24.27351146 T_0(u) + 0.02625669614 T_1(u) - 0.00001435112187 T_2(u) + 0.1304579722 \cdot 10^{-7} T_3(u)) - (5.060831165 T_0(v) + 0.03245494687 T_1(v) - 0.0001866545327 T_2(v) + 0.1630827199 \cdot 10^{-5} T_3(v))$$

By solving the above system will have the solution:  $(x_1 = 17573.35567, x_2 = 2267.637756, y_1 = 6483.69165,$

$y_2 = 43.85811217$ ).

On the computation of values of functions we will have:  $y_1 = 6481.594571$  and

$y_2 = 43.84302918$ , therefore the changes will be respectively for  $y_1$  (-2.097082), while for  $y_2$  (-0.01508299).

**Table 2:** Table of approximations

Approximation by	Food voluminous $x_1$	the concentrate $x_2$	from the solution the weight of the		Estimated value of		changes for	
			milk $y_1$	Beef $y_2$	$y_1$	$y_2$	$y_1$	$y_2$
Chebyshev	17573.35567	2267.637756	6483.69165	43.85811217	6481.594571	43.84302918	-2.097082	-0.01508299

We note by the approaches that  $y_1 \in [2000, 10000]$  and  $y_2 \in [20, 60]$ . For this reason, the options for the computation of the exact solutions through the MAPLE program are:

$x_1 \in [10000, 20000]$  and  $x_2 \in [1000, 5000]$

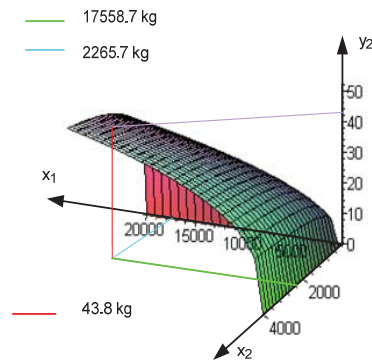
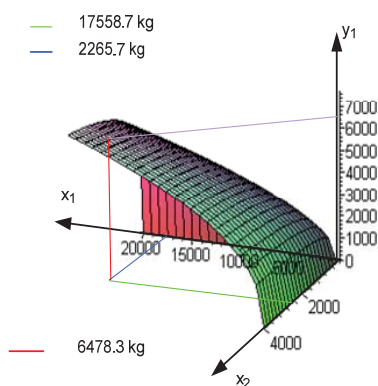
Exact solutions are  $x_1 = 17558.67832$ ,  $x_2 = 2265.746088$ ,  $y_1 = 6478.260594$ ,  $y_2 = 43.82293176$ . It is proved that the maximum of the profit and the minimum of the cost are achieved for  $x_1 = 17558.7$  kg voluminous food and  $x_2 = 2265.7$  kg concentrate. In this case the amount of milk produced by a cow is 6478.3 kg while the average weight of calves at birth is 43.8 kg.

The quantity of fresh food = 10535.22kg,

The amount of dry food = 7023.48kg,

The amount of concentrate = 2265.7 kg.

The cost is 108185.2 L and the revenues are 177287.5 L for a cow in a year. So the profit from one cow is 69102.32 in a year.



Presentation of milk production function and the average weight of calves at birth

$y_1 = 23.31274017 \cdot x_1^{0.424} \cdot x_2^{0.192}$  dhe  $y_2 = 0.3201390009 \cdot x_1^{0.331} \cdot x_2^{0.218}$

## 6. Conclusions

By the study reached the following conclusions:

- During the decision-making process is becoming increasingly evident need for conducting a detailed scientific analysis. Therefore the realization of the livestock production necessarily requires the analysis of the use of the inputs in production.
- Implement the Cobb-Douglas production functions creates the opportunities for the economic analysis of the farm of the
- The study confirmed that the average production levels (21.25115 kg of milk per day) the optimal structure would be: 59% fresh foods, 31% dry foods and 10% concentrates.
- In the general case is shown that the maximum of the revenue is achieved for the same amount of inputs that achieve the maximum of the profit.
- Finally, the conditions of our country, would be preferable voluminous system of nutrition.

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