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Involvement of Algebraic-Geometrical Duality in Shaping Fraction's Meaning and Calculation Strategies with Fractions

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Abstract

Many mathematical concepts and processes, besides the algebraic form of their presentation, can be modeled as well geometrically through diagrams and graphics. Both these aspects of concepts demonstration (algebraic and geometrical aspect) are present on mathematical textbooks of pre-university education. In this paper we consider algebraic and geometrical aspect on 6th grade math textbooks and in particular algebraic-geometrical duality, aiming that the fraction concept and the fraction calculation strategy to be assimilated better by the students. A study was made with 78 students to understand their abilities to express using algebraic symbols and to introduce mathematical situations with geometrical models for "Fractions" chapter. After the analysis of calculative strategies applied by students, in the article it is suggested that algebraic-geometrical duality must be included in teaching based on a complete framework. This will enable students to fully realize the deep understanding of concepts and the calculative strategies they are using.

Keywords: *teaching, mathematic, dual treatment, duality, secondary education*

1. Introduction

Piaget describes three forms of abstraction which are; the empirical abstraction where the focus is on the objects properties and "knowledge on the object derives from his own properties" (Beth & Piaget, 1966, p. 188-189); the pseudo-empirical abstraction where the focus is on actions which "leaves out the properties that the action of the subjects have introduced into objects" (Piaget, 1985, p. 18-19) and reflective abstraction where further constructions can then be accomplished by it using existing structures to construct new structures (Piaget, 1972). The focus of Gray et al., (1999) on perception, action and reflection is consistent with Piaget's three notions of abstraction, with the additional observation that reflective abstraction has a form which focuses on objects and their properties, as well as one which focuses on actions and their encapsulation as objects. Dubinsky et al., (2005) treat APOS (Action, Process, Object, Schema) theory, in which actions are physical or mental transformations on objects. Actions, processes and objects are identified in a schema. Learning of mathematics is a complicated interplay of operational and structural aspects of mathematical concepts. Cognitive activities in math can be different from one student to another, including students such as those ones orientated by images and intuition (conceptual understanding) and those orientated logically, manipulating with symbols without access to their meaning (procedural understanding). Students usually learn routine procedures in a repetitive way. This leads to a misunderstanding of mathematical symbols (Byrnes & Wasik, 1991). Conceptual and procedural knowledge may not develop in independent ways.

Fractions are an important aspect of the elementary curriculum (Hansen et al., 2016) and meanwhile they are well-known to be the most difficult area of mathematics covered in elementary school (Smith, 2002). Those constitute a stumbling block for students of elementary education (Charalambous & Pitta-Pantazi, 2007). Fractions have been used for centuries and are manipulated in a great variety of everyday life situations and in mathematics. Thus why is it so hard for pupils to learn and represent fractions (Gabriel et al., 2013)? Various models have been proposed in order to explain those difficulties (Behr et al. 1983; Mamede et al., 2005; Grégorie, 2008). In this article, we will try to shed light on one situation that leads at difficulties students when they learn fractions. Students' difficulties with fractions often stem from a lack of conceptual understanding. A student might have the procedural knowledge to solve problems with fraction, but might lack conceptual knowledge regarding why this procedure is mathematically justified. Conceptual understanding is difficult to acquire, but it is vital for ensuring e deep and enduring understanding of fractions (Fazio & Siegler, 2011).

Considering the work of the aforementioned authors and orientated by the experience in the implementation of dual treatment in teaching (Gjoci & Kërënxhi, 2013, 2014; Kërënxhi & Gjoci, 2013, 2014, 2015a, b), we studied students' difficulties in learning fractions. Two main components were considered: geometrical treatment (a conceptual component) and algebraic treatment (a procedural component) in shaping fractions meaning and mixed numbers. Our study took place in two different phases: (a) analysis of the textbook "Mathematic 6" (Tato et al., 2012) and (b) test with pupils. In addition to the information on study results, it is suggested the necessity of geometrical modeling of algebraic processes for the development of deep understandings of concepts and fractions calculative strategies as well as with mixed numbers.

2. Textbooks Review

Math's program of 6th grade at elementary education starts with "Fractions" chapter. This chapter includes 20 class hours taking over the 17% of math's program. Analyzing the textbook "Mathematic 6" (Tato et al., 2012), in the "Fractions" chapter we were focused on the algebraic- geometrical presentation of concepts and calculative strategies. The concepts: ratio, fraction and mixed numbers in the textbook "Mathematic 6" (Tato et al., 2012) is shaped starting form geometrical objects through models which are later passed to respective symbols. On the same way it is proceed even with the process of calculation. Therefore, the addition and subtraction of two fractions and mixed numbers, multiplication and division of a fraction with a number are shown with geometrical objects (figures) and later the attention is shifted from the object to the physical world oriented to the manipulation with fraction symbols and mixed numbers' symbols. Let's give some examples of the textbook "Mathematic 6" (Tato et al., 2012).

Fig. 1 is taken from the topic: "The meaning of fractions. Equivalent fractions" (Tato et al., 2012, p. 16). The meaning of fractions in this topic is fulfilled by taking examples which include figures and geometrical models. Exercises include questions such as a part of a circle or triangle should be shown with the equal fraction. We notice that there is no exercise which asks students to draw a figure (model) where the shaded parts to be the same as the given fraction.

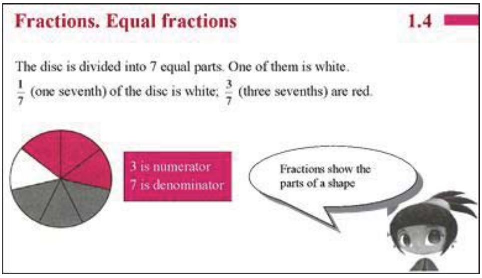

Fig. 1 The algebraic-geometrical presentation of the fraction's meaning

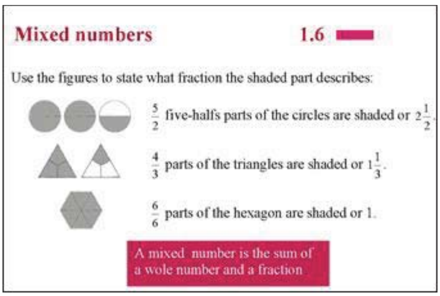

Fig. 2 The algebraic-geometrical presentation of the mixed numbers

Fig. 2 is taken from the topic "Mixed numbers" (Tato et al., 2012, p. 20). Students should express the shaded parts of given model with an improper fraction and after that they should express each improper fraction as a mixed number. In order to fulfill the inverse process, so expressing a mixed number as an improper fraction, the textbook also suggests that students should draw a model. Example: “Express $2\frac{1}{4}$ as an improper fraction. Draw a model to illustrate:

Fig. 3 The model indicates that $2\frac{1}{4}=\frac{9}{4}$

The model (Fig. 3) illustrates that $2\frac{1}{4}$ is composed of 9 quarts or $\frac{9}{4}$.

Than we can make: $2\frac{1}{4}=\frac{(2\cdot 4)+1}{4}=\frac{9}{4}$” (Tato et al., 2012, p. 20).

In order to transmit knowledge over addition and subtraction of fractions with the same denominator, lesson teaching starts with the general rule how these calculations should be made. The given rule should be explained by an example where actions are made with fractions (algebraic shape) and model (geometrical shape). This situation is shown in Fig. 4 (Tato et al., 2012, p. 24). Calculative situations of two fractions multiplication and fraction division by whole numbers starts with geometrical model and after that it goes on with calculation processes with the given fractions. Fig. 5 (Tato et al., 2012, p. 38) and Fig. 6 (Tato et al., 2012, p. 40) show these examples.

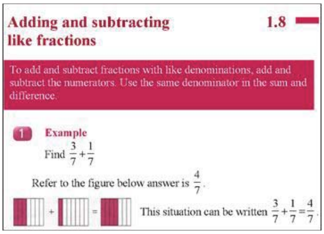

Fig. 4 The algebraic-geometrical presentation of addition of fractions with like denominator

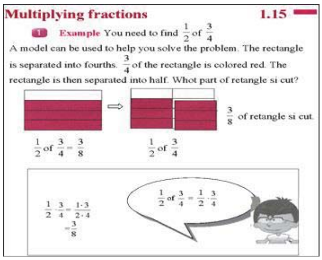

Fig. 5 The algebraic-geometrical presentation of multiplication of two fractions

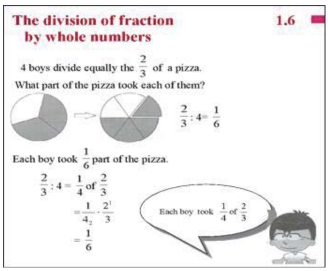

Fig. 6 The algebraic-geometrical presentation of division of fraction by whole numbers

These facts and other analogical ones show that the "Mathematic 6" textbook creates the concept over fraction, mixed numbers and teaches students to make calculations with the numbers following two parallel ways, algebraic and geometrical model. We notice that in the textbook the examples which orientate students to start from the geometrical model and finish with algebraic representation take a huge volume than those of the examples which orientate students to start from the algebraic representation and to finish with the geometrical model. We are interested to know how able are 6th grade students to pass from one situation to another one, so the research questions are:

- Which is the rapport between algebraic and geometrical treatment?
- In what degree are dual treatments applied by the students?
- Is there a connection between algebraic treatment and geometrical treatment?

3. Method

In this article, in the above section we briefly presented some pieces of dual algebraic- geometrical treatment of concepts and some calculative strategies and now we will go on the second phase of the study, test structure and data elaboration of students' tests.

3.1 Participants

In the study were included 78 students of the 6th grade in elementary education. Students' distribution according to school and gender is shown at the Table 1.

Table 1: Students' distribution according to school and gender

	1st school	2nd school	3rd school	4th school	5th school	6th school	Total
Female	6	8	6	8	10	5	43
Male	5	6	4	10	5	5	35
Total	11	14	10	18	15	10	78

3.2 The questionnaire

For the participants in the study was applied a test with questions from "Fractions" chapter. The test included 20 items according to ten categories:

1. Fraction meaning (FM);
2. Comparison of fractions with the same denominator (CFSD);
3. Comparison of fractions with different denominator (CFDD);
4. Addition of fractions with like denominators (AFLD);
5. Addition of fraction with unlike denominators (AFUD);
6. Subtraction of fraction with like denominators (SFLD);
7. Subtraction of fraction with unlike denominators (SFUD);
8. Connection between mixed number and improper fraction (CMNIF);
9. Problem solving in order to express the part of the whole (PSEPW);
10. Problem solving evaluating the whole when it is given a part (PSEWP).

For each category questions were divided in two versions, algebraic treatment (A) and geometrical treatment (G). By algebraic treatment we understand processes expressed with algebraic symbols, as with geometrical treatment we understand processes expressed with geometrical models.

Cronbach's alpha credibility coefficient for the 20 items was .87. As Cronbach's alpha for each item varied from .85 to .87, all items were evaluated as acceptable. Activities which were fulfilled by students for all 20 items are shown in Table 2. In the table issues are grouped according to dual treatment: dual interpretation, dual analyses, dual solution (Gjoci & Kërënxhi, 2013, 2014; Kërënxhi & Gjoci, 2013, 2015a, b) and 10 categories.

Table 2: Evaluation of results over successful thinking

Dual treatment	Categories	Description of students' activity:
Dual interpretation	(1) FM (8) CMNIF	Illustrate the fraction with a diagram; use fraction meaning to express quantity for the shown geometrical model; mixed numbers are transformed into an improper fraction; interpretation of shown geometrical model with a mixed number.
Dual analyses	(2) CFSD (3) CFDD	Compare of fractions with the same denominator; compare shaded parts with two geometrical models when they are the same and express the situation through a numerical inequality; compare of fractions with different denominator; compare shaded parts with geometrical models when the number of its separation is different and express the situation with numerical inequality.
Dual solution Dual formulation	(4/6) AFLD/SFLD (5/7) AFUD/SFUD (9) PSEPW (10) PSEWP	Add/Subtract fractions with like (unlike) denominators; it is given a geometrical model to show the given addition/subtraction of two fractions with like (unlike) denominators. Solve the problem using the algebraic and geometrical strategy; give original strategies through geometrical models.

In order to understand the connection between algebraic treatment and geometrical treatment, were evaluated the frequencies, percentage, averages and standard deviation for each issue. The data are shown on Table 3.

Table 3: Frequencies, averages and standard deviations for each correct answer

Categories	Varieties	Correct answer	M	SD
(1) FM	Algebraic Geometrical	46 (59%) 33 (42%)	.59 .42	.495 .497
(2) CFSD	Algebraic Geometrical	62 (79%) 37 (47%)	.79 .47	.406 .503
(3) CFDD	Algebraic Geometrical	44 (56%) 29 (37%)	.56 .37	.499 .486
(4) AFLD	Algebraic Geometrical	64 (82%) 23 (29%)	.82 .29	.386 .459
(5) AFUD	Algebraic Geometrical	56 (72%) 13 (17%)	.72 .17	.453 .375
(6) SFLD	Algebraic Geometrical	67 (86%) 26 (33%)	.86 .33	.350 .474
(7) SFUD	Algebraic Geometrical	42 (54%) 15 (19%)	.54 .19	.502 .397
(8) CMNIF	Algebraic Geometrical	34 (44%) 27 (35%)	.44 .35	.499 .479
(9) PSEPW	Algebraic Geometrical	31 (40%) 15 (19%)	.40 .19	.493 .397
(10) PSEWP	Algebraic Geometrical	18 (23%) 10 (13%)	.23 .13	.424 .336

In order to understand how many students are able to implement dual algebraic-geometrical treatment, this data is shown in the Tables 4, 5, 6.

Table 4: Algebraic, geometrical, and dual algebraic-geometrical interpretations of students

Categories	Students who interpret only through algebraic shape	Students who interpret only through geometrical shape	Students who interpret through algebraic-geometrical duality	No answer	Total
(1) FM	18 (23%)	5 (6%)	28 (36%)	27 (35%)	78 (100%)
(8) CMNIF	15 (19%)	8 (10%)	19 (25%)	36 (46%)	78 (100%)

Table 5: Algebraic, geometrical, and dual algebraic-geometrical analyses of students

Categories	Students who analyze only through algebraic shape	Students who analyze only through geometrical shape	Students who analyze through algebraic-geometrical duality	No answer	Total
(2) CFSD	30 (39%)	5 (6%)	32 (41%)	11 (14%)	78 (100%)
(3) CFDD	23 (30%)	8 (10%)	21 (27%)	26 (33%)	78 (100%)

Table 6: Algebraic, geometrical, and dual algebraic-geometrical solutions of students

Categories	Students who solve only through algebraic shape	Students who solve only through geometrical shape	Students who solve through algebraic-geometrical duality	No answer	Total
(4) AFLD	43 (55%)	2 (3%)	21 (27%)	12 (15%)	78 (100%)
(5) AFUD	44 (56%)	1 (1%)	12 (16%)	21 (27%)	78 (100%)
(6) SFLD	42 (54%)	1 (1%)	25 (32%)	10 (13%)	78 (100%)
(7) SFUD	27 (35%)	0 (0%)	15 (19%)	36 (46%)	78 (100%)
(9) PSEPW	18 (23%)	1 (1%)	13 (17%)	46 (59%)	78 (100%)
(10) PSEP	11 (14%)	3 (4%)	7 (9%)	57 (73%)	78 (100%)

An inter-correlation between issues was investigated using Pearson Product-Moment Correlation in order to understand if there exists any relationship between algebraic and geometrical treatment. Data are shown in the Table 7.

Table 7: Inter-correlation coeficients betwen items

Measures	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)	(20)
(1) FM –A	1																			
(2) FM-GJ	.450**	1																		
(3) CFSD-A	.286*	.242*	1																	
(4) CFSD-GJ	.062	.278*	.228*	1																
(5) CFDD-A	.108	.229*	.450*	.369**	1															
(6) CFDD-GJ	.156	.361**	.259*	.248*	.597**	1														
(7) AFLD-A	.085	.198	.093	.195	.244*	.152	1													
(8) AFLD-GJ	.196	.072	.189	.228*	.399**	.433**	.156	1												
(9) AFUD-A	.230*	.248*	.246*	.483**	.310*	.187	.449**	.093	1											
(10) AFUD-GJ	.053	.174	.227*	.324**	.195	.368**	.120	.204	.541**	1										
(11) SFLD-A	-.038	.123	.342*	.164	.237*	.235*	.098	.237*	.100	.181	1									
(12) SFLD-GJ	.313**	.110	.157	.128	.418**	.469**	-.024	.201	.557**	.341**	.208	1								
(13) SFUD-A	.117	.220	.167	.483**	.313**	.287*	.304*	.563**	.317**	.414**	.364**	.327**	1							
(14) SFUD-GJ	.209	.109	.167	.232*	.318**	.567**	.143	.161	.469**	.655**	.198	.452**	.552**	1						
(15) CMNIF-A	.050	.189	.319**	.199	.097	.180	-.060	-.081	.112	.162	.059	.140	.091	.293*	1					
(16) CMNIF-GJ	.059	.086	.169	.259*	.496**	.165	.200	.151	.180	.036	.217	.295**	.286*	.260)	.393**	1				
(17) PSEPW-A	.092	.206	.088	.291*	.435**	.405**	.175	.334**	.222	.269*	.179	.437**	.537**	.534**	.237*	.400**	1			
(18) PSEPW-GJ	.209	.109	.087	.101	.188	.230*	.143	.089	.327**	.306**	.083	.256*	.414**	.505**	.293**	.124	.468**	1		
(19) PSEWP-A	.271*	.147	.128	.297*	.272*	.586**	.098	.276*	.313**	.408**	.135	.385**	.452**	.582**	.194	.177	.426**	.428**	1	
(20) PSEWP-GJ	.164	-.018	.005	-.050	.096	.102	.179	.070	.988	.240*	-.065	.201	.298*	.299**	.127	.124	.315**	.427**	.591**	1

*p<0.05, **p<0.01

4. Discussions and Conclusions

Analyzing the results of Table 3, we notice that the algebraic treatment and geometrical treatment rapport of 10 studied issues are on algebraic treatment part. Such a result more or less was expected for the "Fractions" chapters in the textbook "Mathematic 6" (Tato et al., 2012). This happens because for concept shaping and on calculative processes, the textbook starts from geometrical models using geometrical models to continue with symbols and algebraic operations. After this, geometrical models are not asked anymore and in the textbook materials' exercises we have no cases of students to whom are asked to use geometrical models in their answers. Referring to Tables 4-6 we can say that approximately one in three students interprets through algebraic-geometrical duality (Table 4), one in three analyses through algebraic-geometrical duality (Table 5), one in four students solves the exercises through algebraic-geometrical duality (Table 6) and one in seven students solves problems through algebraic-geometrical duality (Table 6). These results are good ones by comparing the fact that neither the text nor the teachers pay attention to dual algebraic-geometrical treatment. By studying Table 7 it is noticed that in fraction meaning forming exists a relationship strongly considerate between algebraic and geometrical interpretation of fractions meaning. 20% of students have a correct concept over fractions. They can show by symbols a part of the given model, they can reveal the position of a point on a number line which is expressed by fraction and are able to express through a geometrical model one part of the same figure related to the given fraction. In Table 7 we can see that there exists the strong relationship between algebraic and geometrical interpretation of mixed numbers mining. The strong relationship exists also between algebraic and geometrical solutions of exercises with addition (subtraction) of fractions where the fractions have unlike denominators and between algebraic and geometrical solutions of problems.

We came to the conclusion that besides the fact that teaching begins with geometrical models in order to make clear the concept of refraction and arithmetical operations, than we have to deal with concepts and arithmetical operations the ones that dominate on students through which they operate "mechanically" by "forgetting" geometrical models. We think that algebraic- geometrical duality should be included in the teaching process into a complete framework. This will enable students to reach deep understanding of concepts and calculative strategies to the ones they are using. Through an analogical way, just as it was done in the algebraic treatment and geometrical models and refraction operations, can be done algebraic treatment and geometrical models and decimal numbers, percentages and equations and their systems.

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