**Research Article**



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# Mathematical Expectation: Conceptual Difficulties Among Students

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***Abstract***

*In this paper, we are interested in the introductory teaching of the notion of Mathematical expectation, also known as the expected value. Our research hypothesis is that teaching based on a formal approach may be a didactic failure. The long-term observations of a reduced number of students made it possible to identify various difficulties and conceptual obstacles around the interpretation and the implementation of this notion. The difficulties that obstruct the availability of this object are due mainly to conceptual confusion between the notion of expected value and the notion of probability, others are due to erroneous representations. A quantitative analysis of students' productions showed that the relevance of the formal approach to the interpretation of expected value had no effect on its validity, compared to the intuitive approach which had an impact on its validity. These results thus encourage the adoption of a dialectic: formalism / intuition in the introduction of probabilistic formalism.*

***Keywords:*** *expected value, probability distribution, formal approach, intuitive approach*

## 1. Introduction and Problem

The teaching of the notion of mathematical expectation, as it is commonly practiced in higher education, is often conducted based on formal definitions, accompanied by applications highlighting calculations of formal aspects. Thus, probabilistic modeling questions rarely represent the essential task of student activity. These modeling activities are of course the most difficult (if not the most delicate) to manage with students in a tutorial-type activity, so they are often taken care of largely by the teacher himself.

In its evolution, the notion of mathematical expectation has undergone the succession of two approaches. First, Pascal and Fermat (1654) spoke of "equal chance" (Pichard, 2001); to express the notion of mathematical expectation. Secondly, Christian Huygens (1656) managed to overcome the condition of equiprobabilities adopted by the two founders of modern probability Pascal and Fermat. Indeed, Christian Huygens gives the following definition (Huygens, 1657): ≪ *Having p chances of obtaining a and q chances of obtaining b, the chances being equivalent, gives me* $\frac{pa+qb}{p+q}$ », at the time, the notion of mathematical expectation played an essential role in overcoming the epistemological difficulties of probabilistic modeling.

In general, modeling involves considering the relationships between quantified variables, mathematical procedures, and concepts that will all be used at the same time to arrive at an abstract result. The latter will then be interpreted and evaluated in the context studied (Savard, 2008). In this regard, Michel Henry (2001) showed that the role of "concrete" mathematical modeling currently tends to highlight the instrumental character of mathematics. In some international studies (KAHANE, 1986), mathematics finds its legitimacy as a service discipline, through the transfer of its concepts and models to solve external problems, posed by the development of knowledge in other sectors of human activity (Henry, 2001). In this regard, thanks to his axiomatic approach, Kolmogorov (1933) clearly formalized the concept of "probability" by keeping the link between the notion of probability and reality in which intuition plays a central role; this is what Kolmogorov confirms in his work by saying « Every axiomatic (abstract) theory admits, as is well known, of an unlimited number of concrete interpretations besides those from which it was derived. Thus, we find applications in fields of science which have no relation to the concepts of a random event and of probability in the precise meaning of these words » (Komogorov, 1933).

Furthermore, the elementary concepts of probability theory; axiomatic probability, random variable, law of probability, etc.; pose the didactic problem of the distance between model and reality, and make the modeling process difficult for students to assimilate. They are not perfectly comfortable using abstract knowledge of probabilities and models of random variables to solve concrete problems. In this regard, Bernard DANTAL (2001) worked on the question: "Is the observation of reality necessary to build a model? ". This question seems to us to be a central element of didactic research about probabilistic modeling through mathematical expectation.

Several works, notably those of Rouan (1990), Zaki (1991 and 1992), Gras and Totohasina (1995), Tamimi (1995) and Henry (2001, 2008, 2009 and 2011), Amrani and Zaki (2015), have noted various difficulties and conceptual confusions in probability, particularly at the university level, regarding the notions of random variables and the law of probability:

- The distance between "random variable" and "probability law"; students talk about the law of probability without defining the corresponding random variable;
- The distance between probabilistic models and their interpretations in real situations;
- Difficulties in identifying a suitable model in a random experiment;
- Difficulties in identifying the characteristics of probability laws, with reduction of probability laws to simple mathematical formulas; this is the case for example of the binomial law which is often associated with the simple expression $p(X=k)=C_n^k p^k (1-p)^{(n-k)}$ without specifying the conditions for the implementation of this law.
- Difficulties in the formal aspect of the notion of the random variable as a modeling tool in

probability theory.

For his part, Michel HENRY (2001) introduced the following stages regarding probabilistic modeling: The first stage of modeling at the level of the concrete situation is based on the observation of a real situation and its description in current terms, the second step concerns the translation of the concrete situation into formal terms and the third step must be devoted to returning to the question asked to translate the mathematical results obtained into the terms of the pseudo-concrete model.

The problem of this research concerns the identification and analysis of the difficulties encountered by students when faced with the notion of mathematical expectation. Thus, we question the very formal approach, at university, to the teaching of this notion, the didactic consequences of which allow us to formulate the following two research hypotheses:

**First research hypothesis:** The probabilistic formalism of the notion of mathematical expectation is difficult to understand and interpret.

**Second research hypothesis:** The lack of reference, or even the absence of probabilistic situations in university teaching, using random experiments favoring an intuitive approach to the notion of mathematical expectation (Girard, 2001), hinders a good understanding by students, and therefore a good appropriation of the probabilistic formalism (Greer, 2001).

In the following paragraph, we will see that the epistemological analysis of the genesis of this notion largely corroborates these two hypotheses.

The mathematical expectation dates back to a correspondence between Pierre de Fermat and Blaise Pascal in 1654. Their inspiration came from a problem about games of chance, proposed by the Chevalier de Méré. De Méré inquired about the proper division of the stakes when a game of chance is interrupted. Suppose two players, A and B, are playing a three-point game, each having wagered 32 pistoles, and are interrupted after A has two points and B has one. How much should each receive?

Pascal estimated the player's expected gain through a recurring approach while Fermat treated the problem in a combinatorial manner. This was a starting point for the emergence of Pascal's famous arithmetic triangle as well as the emergence of his famous recurrence relation $C_{n-1}^{k-1} + C_{n-1}^{k} = C_{n}^{k}$

During his treatises, Pascal spoke neither of probability nor of expectation of gain but his intuitive idea remains to associate a gain with a chance of obtaining it (Trotignon, 1998).

Furthermore, the treatment of these problems allowed Huygens to correctly construct algebraic formulas allowing the calculation of the value of “Expectation”. For Huygens if he hopes a or b, and that one or the other can happen to me as easily, we must say that his expected value is worth $\frac{a+b}{2}$.

In 1657, Christian Huygens, in his book “On calculation in games of chance” (Huygens, 1897), generalized the method of calculating the expected gain by going beyond the condition that the game be fair. For him, if we have p chances of winning sum a and q chances of winning sum b then for the game to be fair, we must bet $S = \frac{ap+bq}{p+q}$. In other words, if we have p chances of winning sum a and q chances of winning sum b then the player's expected gain is $S = \frac{ap+bq}{p+q}$. Thus, he clearly formalized the notion of mathematical expectation, which he called the value of my luck. Furthermore, this notion is popularized by Christian Huygens in his Treatise on Chance under the name of “value of luck”. Then, mathematical expectation is extended to other areas of their daily life, with his brother, they became interested in life expectancy (Bellos, 2011).

Then, this mathematization of the value of the mathematical expectation of a game of chance situation was one of the central factors in understanding how Jacques Bernoulli developed one of the beginnings of the mathematics of probability resulting from the problem of estimating probabilities.

Laplace (1749-1827), for his part, carried out a synthesis of all the knowledge on probabilities of his time: he has the merit of having made an important contribution to the development of probabilities, comparable to that of Euclid for geometry. Although he remained in the context of games of chance, he managed to define expectation more elaborately than that of Huygens, and more directly, because it does not refer to the notion of a fair game. For him, the expectation « *generally* *expresses the advantage of the one who expects any good in suppositions which are only probable. This advantage, in the theory of chance, is the product of the expected sum by the probability of obtaining it: it is the partial sum which must return when one does not want to run the risks of the event, supposing that the distribution is done proportionally to the probabilities (…) When the advantage depends on several events, it is obtained by taking the sum of the products of the probability of each event by the good attached to its arrival*» Laplace (p. XIX).

It is clear that in his definition of mathematical expectation, Laplace does not explicitly define the notions of random variable and the law of probability, nevertheless, all the mathematical ingredients of these notions appear in this definition; moreover, the reasoning that Laplace uses, for example, in the treatment of the problem of the "tournament" or even that of the "parties", easily lends itself to a treatment using the notion of random variable.

During this initial period of probability theory, we note that the treatment of chance problems was essentially done in a gaming context, with problems referring to winning and the expectation of winning: in this type of context, these notions remain intuitive and natural. Thus, the treatment of these notions was done intuitively, without explicitly requiring formal mathematical definitions of the notions of random variables and the law of probability. This certainly also contributed to the difficulty in the immediate emergence of the formalism of probability theory.

In general, the value of the mathematical expectation, in countable cases, represents the barycenter of the values taken by a random variable weighted by the probability of its occurrence. In the case where a criterion takes non-countable values, this criterion has a probability density. In this regard, Laplace was the first to define the probability density of a so-called continuous random variable, whose mathematical expectation is the integral, over $\mathbb{R}$, of these values taken by the random variable (player's gain) weighted by their image by density. In addition, there is not always a mathematical expectation, in particular, that of long-tail distributions, such as the Cauchy distribution, which have non-convergent mathematical expectations, so no mathematical expectations.

On the other hand, expectation is an important characteristic of a probability law, it plays a role as a position indicator. Thus, a random variable is said to be centered if its expectation is zero.

Furthermore, and, sometimes, the complexity of probabilistic modeling directs us to adopt the law of large numbers to calculate the mathematical expectation through the repetition of random experiments of the same type a large number of times. For example, during a single roll of the dice, which is not necessarily balanced, each time the player wins the complement of the result compared to 6, that is to say: if the result is 1 the player will win 5, … and if the result is 6 then he will win 0. Therefore, it is difficult to predict the player's average gain on a reduced number of throws of the die. But, by throwing the die a large number of times (for example ten thousand times), then if the die is balanced the average of the player's gain approaches (oscillates around), 2.5 which will be the mathematical expectation of the player's gain. So, you will see below the simulation of the player's average gain (Figure 1):

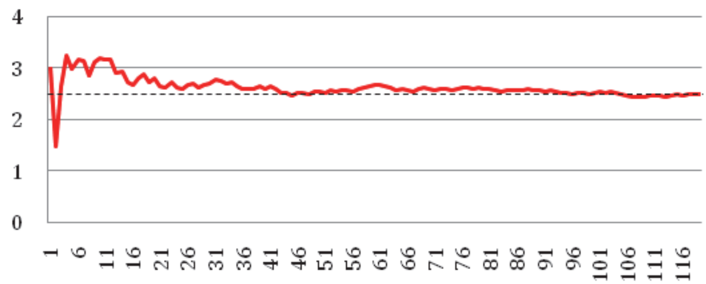


**Figure 1:** Average player gain until the ith throw

In addition, Expectation and the law of large numbers also make it possible to invalidate a law of probability. It is said that Henri Poincare used it, along with other clues, to highlight the dishonesty of his baker (BELLOS, 2011). Indeed, the weight of a loaf is subject to random fluctuations but its expectation is fixed by law. The weight of a loaf advertised at 1 kg, for example, can fluctuate around this value. Poincare would have weighed the bread purchased from his baker over a long period and would have found that its average weight was well below 1 kg. This average was too far from expectation and indicated embezzlement on the part of the merchant. This implies that the law adopted the first time is wrong. The frequency of a value is, in general, an average; its limit, by repeating the experiment concerned a large number of times, is the mathematical expectation.

This epistemological analysis shows the great importance of the contribution of the intuitive field to the emergence of the notion of mathematical expectation in its formal aspect, because indeed the notion of mathematical expectation is intimately linked to concrete situations where the frequentist approach takes place. This strongly supports the two research hypotheses of our problem; teaching that ignores the intuitive aspects and only favors the formal aspect of these notions will generally be doomed to didactic failure. Thus, unlike teaching models which rely more on intuitive representations, teaching based on theoretical presuppositions cannot always guarantee the availability of the notions of random variables and the law of probability (Lecoutre and Fischbein, 1998). The results of our experiment conducted with students will show the validity of our hypotheses.

## 2. Experimentation: Methodological Approach and Questionnaire

Our problem initially results from effective observations in the field regarding probabilistic difficulties encountered by our students training at the university. Furthermore, the epistemological study allowed us to make an initial analysis of the origin of these difficulties and to establish our research hypotheses. We, therefore, favored an exploratory approach during our research to properly identify and better analyze these difficulties. We then used a questionnaire, in which we questioned the students on different aspects relating to the notions of random variable and the law of probability, their natures, on the meaning of the parameters of a random variable (mathematical expectation), and their interpretation; going so far as to explore the availability of these notions in a situation of elementary probabilistic modeling.

In this study we will limit ourselves only to the analysis of the part of the questionnaire relating to aspects concerning the notion of mathematical expectation: this part, in itself, is already very rich in information regarding to our general problem of probabilistic modeling, and in particular about the difficulties and conceptual representations of students faced with the status of a mathematical expectation of a random variable.

Individually, the students completed the questionnaire over an average duration of two hours. The sample of 25 students interviewed, all volunteers, was taken from the population of students who validated the probability module in the third semester of the basic BA cycle (composed of six semesters). The choice of students who have validated the probability module is not trivial. Indeed, to better support the results of our analyses, and while remaining consistent with our research hypotheses, we seek to analyze the impact of formal teaching about random variables on the conceptual representations of mathematical expectation, of students who have validated said probability module.

Finally, to situate the teaching content of this module, we will specify that the latter is established synthetically from probabilized spaces introduced through the axiomatic definition of a probability, random variables as measurable functions defined on probabilized spaces, probability laws as image probabilities, classic definitions of the parameters of random variables and their properties, with a large part devoted to the study of the usual discrete and continuous probability laws.

The analysis of the students' productions was carried out according to a qualitative procedure, with a view to obtaining a classification of the conceptual representations identified. Indeed, the methodological approach is exploratory, with a reduced number of students (25 students), within return individual observations of relatively long durations; it was relevant to qualitatively analyze the students' responses one by one (M. Zaki, 2004), looking for common conceptual elements that would allow a classification of the students' conceptual representations: thus, the analysis of the productions will be developed in the form of a classification of students' conceptual representations about the notion of mathematical expectation.

## 3. Conceptual Representations of the Law of Probability among Students

The notion of the law of probabilities of a random variable poses a real problem for students; only 32% of those questioned formulated a correct definition of this notion. The majority of them, 75%, refer to formal definitions of the law of probability relating to image probability $p_X$, which verifies the axioms of probability, such that any part $I$ in $\mathbb{R}$, interval or countable union of intervals of $\mathbb{R}$, we have: $p_X(I) = p(X \in I) = p\left(X^{-1}(I)\right) = p\left(\{w \in \Omega / X(w) \in I\}\right)$.

However, 25% of them adopt an intuitive approach to the definition of a probability law relating to the probabilities of the values taken by the random variable.

On the other hand, a minority of 4% of students defined the probability law using the mass function and the distribution function, in a very confusing manner.

The rest of the students (28%) reduced the notion of the law of probability to only calculations of probabilities of the values taken by the random variable, even to the results of the random experiment, or even to the probabilities of events. Otherwise, 36% of the students surveyed provided incorrect answers or remained unanswered.

In conclusion, a strong majority (75%) adopted a formal approach to the definition of the law of probability, which is not surprising for university students as well because the law of probability is difficult to associate with a real interpretation. However, this majority does not formulate this definition in a relevant way: only one in three students (32%) gives a correct definition. Otherwise, overall, we found that, for the rest of the students, a probability law represents either calculations of probabilities linked to events in relation to the random variable, or quite simply the results of random experiments; otherwise, they have no definition of a law of probability.

The following table (Table 1) summarizes the distribution of responses from the 25 students questioned about the definition of a probability law:

**Table 1:** Approaches to the definition of “probability law”

| Correct | Partially correct | False | No answer |
|---|---|---|---|
| 32% | 4% | 32% | 32% |

Furthermore, the majority (76%) of the students questioned managed to cite at least four usual probability laws. There is therefore a significant gap among students between the relevance of producing the definition of the law of probability and the citing of examples of usual laws of probability: it is easier for them to cite usual laws of probability than to provide a formal definition of a probability law. We summarize in the following table (Table 2), the examples of usual laws and frequencies of their citations:

**Table 2:** Probability laws cited by the 25 students interviewed

| Laws | Uniform | Bernoulli | Binomial | Normal | Hypergeometric | Geométric | Poisson | Others |
|---|---|---|---|---|---|---|---|---|
| **Pourcentage** | 16% | 76% | 92% | 76% | 32% | 28% | 88% | 68% |

Now, what about the availability and recognition of a probability law among students?

## 4. Availability and Recognition of a Law of Probability

Faced with an elementary situation where students are asked to recognize the random variable and the law of probabilities associated with it, to best model the underlying random experiment, only 24% of the students questioned managed to give an interpretation using a random variable and its probability law. The remaining three-quarters of students (76%) all presented difficulties in recognizing and making available such modeling. Among these students, a large majority, or 56% of the students surveyed, gave an interpretation of this situation in terms of event probabilities. The following table (Table 3) gives the distribution of responses from the students interviewed:

**Table 3 :** Availability and recognition of probability laws

| Law of probability | Probabilities of events | Others |
|---|---|---|
| 24% | 56% | 20% |

We have, moreover, sought to further our investigation with students regarding usual probabilistic models, by proposing the following random experiment situations:

*Which random variable or probability law can be used to model the following random experiments:*
*a- "A balanced coin thrown once in the air will land on tails"*
*b- "Throw a balanced die and get "5" on its upper side"*
*c- "Throw a balanced die 100 times and get an ace 20 times"*
*d- "Two people call the same cell phone at the same time"*
*e- "The number of throws necessary to obtain Tails for the first time"*

The following table (Table 4) summarizes the students' correct answers:

**Table 4:** Correct approaches to modeling given situations

| **Situation** | *a* | *b* | *c* | *d* | *e* |
|---|---|---|---|---|---|
| **Correct** | 56% | 28% | 48% | 16% | 12% |
| **Model** | **Bernoulli** $\mathcal{B}\left(\frac{1}{2}\right)$ | **Bernoulli** $\mathcal{B}\left(\frac{1}{6}\right)$ | $\mathcal{B}\left(100;\frac{1}{6}\right)$ | $\mathcal{P}(\lambda)$ | **The geometric law** |

The recognition and availability of Poisson and geometric probability laws are those that are the least successful among students, with 16% and 12% success rates respectively. Now, the probability laws that are best recognized remain the Binomial law with 48% and Bernoulli's law, with 56% for situation "a" and only 28% for situation "b".

Furthermore, this confirms once again the non-availability and non-recognition of the usual model, the most basic, namely that of Bernoulli's law, because barely one student in three manages to use it correctly for modeling of 'an elementary random experiment situation.

In fact, there is once again a large gap between the formal understanding of the usual probabilistic models and their putting into practice, through their availability and recognition, in simple situations of random experiments. The following graph (Figure 2) clearly highlights this gap:

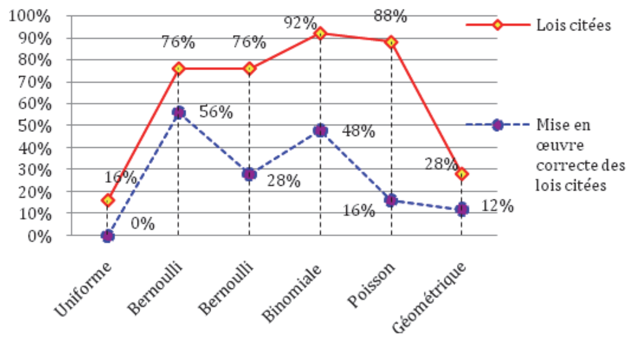


**Figure 2:** Comparison graph between models of usual probability laws cited and their implementations in given probabilistic situations.

The most striking, also, in these differences is that which concerns the Poisson law: this is among the usual laws most cited by students (88%) and remains among the least available and recognized (16%) for the modeling of an elementary random experiment.

## 5. Analysis of Conceptual Representations Versus the Notion of Mathematical Expectation

### *5.1 Analysis of interpretations of the notion of mathematical expectation*

The mathematical expectation theoretically represents the average value taken by the random variable, also called the central tendency of the random variable or the expected value. At the experimental level, the expectation represents the limit of the averages of the values of a quantitative character weighted by their frequency. In other words, the average value of a random variable in a random experiment is an approximation of the value of the mathematical expectation. In general, the mathematical expectation is a real value, which only makes sense in the overall context of the modeled situation. This central value is only of interest if it is compared to the values taken by the random variable: this is why it represents a position parameter.

It often happens that students interpret the special cases of mathematical expectation" $E(X) > 0$ " by "the situation is favorable" and " $E(X) < 0$ " by "the situation is unfavorable" without taking into account the context to which the random variable refers. For this reason, we asked students the question of interpretation of mathematical expectation, as well as that of particular cases "$E(X) > 0$", "$E(X) = 0$" et "$E(X) < 0$".

The first observation is that 24 % of all students surveyed provided an intuitive approach to the interpretation of the mathematical expectation E(X), by interpreting it in terms of the probability of an event. Here are some responses from the students:

- Student E2: « $E(X) = \sum_{i=1}^{n} p_i X_i$ *which represents the parameter which makes it possible to decide whether a game is legal or illegal (if E(X)<0 I have less luck than the other player)* »
- Student E3: « $E(X)$ *is the percentage of chance in an experiment* ».
- Student E6: « *Mathematical expectation means the ratio of carrying out the action, that is to say the degree of achievement or the ratio of probability*».
- Student E12: « *The interpretation of mathematical expectation is the average probability of a random experiment* ».
- Student E17: « *A mathematically obtained result that estimates in advance the probability of a fact occurring* ».
- Student E21: « $E(X)$ *it is the average of the probabilities of the values taken by* $X$ ».

When interpreting the particular cases of expectation $E(X) = 0$, $E(X) > 0$ and $E(X) < 0$, those students interviewed who have already interpreted mathematical expectation as the probability of an event, have further interpreted the mathematical expectation as the probability of an event the particular cases indicated above. Here are some of their responses:

- Student E2 interpreted the situation « $E(X) = 0$ » in terms of probabilities $p_i$ which disperse around $E(X)$ without taking into account the values taken by the random variable.
- Student E3 provided an inspired interpretation of the notion of probability in the following form: « $E(X) = 0$ *: no chance,* $E(X) > 0$ *: there is a chance and* $E(X) < 0$ *: failure* ».
- Student E6 interpreted the case $E(X) = 0$ : the non-performance of the action, that is to say the degree of completion is zero, $E(X) > 0$ : there is a contribution of the completion of the action and $E(X) < 0$ : this case is unrealistic.
- Student E12 interpreted the case "$E(X) = 0$ " by « no mathematical expectation».
- The interpretation of student E10 is: « $E(X) = 0$ : there is no expectation, the data must be revised ».
- Student E17 interpreted the case "$E(X) = 0$" by the fact that $X$ is a centered random variable.
- Student E21 provided an interpretation in terms of probabilities: « " $E(X) = 0$ " there is equal probability, "$E(X) > 0$ " the probability is higher for the values taken by the random variable $X$ and " $E(X) < 0$ " the probability is lower for the values taken by the random variable $X$ ».
- Students E7 and E12 did not interpret the particular cases of mathematical expectation.

The interpretation of a mathematical expectation as a probability of an event appeared among 24 % of students, and 28 % of students interpreted it as such in particular cases, "$E(X) > 0$" and "$E(X) < 0$". The persistence of such an interpretation appears especially among students who have already provided it for the question « What does $X$ represent? ». Among these, 83% interpret particular situations once again "$E(X) = 0$", "$E(X) > 0$" and "$E(X) < 0$" in terms of probabilities.

Thus, the interpretation of a mathematical expectation in terms of probabilities of events seems to be quite anchored among one in four students, which represents a non-negligible proportion of students whose conceptual representation of mathematical expectation is wrong.

Based on this observation, it becomes interesting to ask the question whether the “formal” or “intuitive” nature of students’ approach to the interpretation of mathematical expectation has an effect on its validity.

### 5.2 *Effect of nature (formal or intuitive) in the approach to the interpretation of mathematical expectation on its validity*

Regarding the question of interpreting mathematical expectation, 24% of all students surveyed did not answer. For those who gave an interpretation, 16% took a formal approach, and 84% an intuitive approach. We, therefore, sought to know if the nature of interpretation (intuitive or formal) of $E(X)$ could affect the validity (correct or not correct) of the interpretation of $E(X)$ provided. By discarding the non-responses (6 students), we cross-referenced the responses of the remaining students (19 students), considering the qualitative variable N: “Approach to interpreting mathematical expectation”, assigning it two modalities, “formal » and “intuitive”, with the qualitative variable V: “Validity of the interpretation of the mathematical expectation provided”, by assigning it the two modalities “correct” and “not correct”. We therefore obtained the following table (Table 5) :

**Table 5 :** Crossing variables « Approach to interpreting mathematical expectation » with « validity of the interpretation of the mathematical expectation provided »

| Validity of interpretation of $E(X)$ / Nature of interpretation of $E(X)$ | Correct | Not correct | |
|---|---|---|---|
| Intuitive | 9 | 7 | 16 |
| Formal | 1 | 2 | 3 |
| | 10 | 9 | 19 |

Regardless of the approaches chosen by the students, the previous table (Table 5) shows that correct and incorrect answers are equally distributed. On the other hand, the majority of students adopt an "intuitive" approach in their interpretation (84%), and that this is more favorable to a correct interpretation of the mathematical expectation (9/16 students), compared to those who adopted a formal approach (1/3 students). The hypothesis to put forward in this situation would therefore be H1| "The intuitive approach affects the validity of the interpretation of mathematical expectation". We will test using Fisher's exact test the null hypothesis Ho|, which is the opposite of H1|, and which can be stated in the present case as Ho| "There is independence between the "intuitive" and "formal" approach on the validity of the interpretation of mathematical expectation." By keeping the fixed margins in table 5, and considering the most extreme situations, under the hypothesis Ho|, we obtained the probability:

$$p = \frac{C_{10}^{9}.C_{9}^{7}+C_{10}^{10}.C_{9}^{6}}{C_{19}^{16}} \simeq 0{,}458$$

Fisher's exact test is far from significant and the null hypothesis Ho cannot be rejected. Taking into consideration the value of $p \cong 0{,}458$, we cannot conclude that there is a dependence between the nature of the interpretation of mathematical expectation and the validity of its interpretation. To better support this result, we sought to know if the nature of the "formal" or "intuitive" approach adopted by the students in the general case of mathematical expectation had an effect on the validity of the interpretation of particular cases "$E(X) = 0$", "$E(X) > 0$" and "$E(X) < 0$".

### 5.2.1 *Effect of the nature of the approach (formal or intuitive) in the interpretation of the general case of mathematical expectation on the validity (correct or not correct) of the interpretation of particular cases*

No student managed to give a correct interpretation to the particular cases of mathematical expectation "$E(X) = 0$", "$E(X) > 0$" and "$E(X) < 0$". Only one student out of the 25 students questioned managed to give a partially correct interpretation, and this using an intuitive approach. Furthermore, 20% of students did not answer this question, these are the same students who did not give an answer to the previous question, relating to the general interpretation of mathematical expectation.

Once again excluding non-responses, we crossed the two qualitative variables N: « Nature of the approach to the interpretation of mathematical expectation » with the two modalities « Formal » et « Intuitive », and V: « Validity of the interpretation of the special cases "$E(X) = 0$", "$E(X) > 0$" and "$E(X) < 0$". », by assigning two modalities to it: « correct » and « partially correct ». We therefore obtained the following table (Table 6) :

**Table 6 :** Crossing variables « Nature of the mathematical expectation interpretation approach » and « Validity of the interpretation of particular cases of mathematical expectation $E(X) = 0$ , $E(X) > 0$ and $E(X) < 0$ » among the 20 students interviewed

| Interpretation of particular cases of $E(X)$ / Nature of interpretation of $E(X)$ | Not correct | Partially correct | |
|---|---|---|---|
| Intuitive | 15 | 1 | 16 |
| Formal | 4 | 0 | 4 |
| | 19 | 1 | 20 |

The interpretation approach most widely used by students is "intuitive", and it is the only one that provided a partially correct answer. From there, we can naturally be in favor of hypothesis H1 "The nature of the interpretation approach in the general case of a mathematical expectation has an effect on the validity of interpretation of particular cases of mathematical expectations "$E(X) = 0$", "$E(X) > 0$" and "$E(X) < 0$". ».

Using once again the Fisher exact test, in testing the hypothesis Ho|, representing the opposite of H1|, gives the probability:

$$p = \frac{C_{19}^{15} . C_1^1}{C_{20}^{16}} = 0,8$$

Thus, Fischer's exact test is far from being significant, and the null hypothesis Ho cannot be rejected: the nature of the approaches to the interpretation of E(X) has no effect on the validity of the interpretation of its particular cases. Given the distribution of table 6, we could even specify that an intuitive approach in the general interpretation of mathematical expectation significantly favors the validity of interpretation of particular cases "$E(X) = 0$", "$E(X) > 0$" and "$E(X) < 0$".

*5.2.2 Effect of the relevance of the intuitive approach in the interpretation of the general case of mathematical expectation on the validity of interpretation of particular cases*

Following the conclusion of the previous result, we wanted to push the analysis further, by studying the effect of the relevance of the intuitive approach in the interpretation of the general case of mathematical expectation, on the validity of the interpretation special cases. We therefore crossed the two qualitative variables I: « Intuitive approach in the general interpretation of mathematical expectation», with two modalities « relevant » or « irrelevant », and V: « Validity of the interpretation of particular cases "$E(X) = 0$", "$E(X) > 0$" and "$E(X) < 0$" », with the two modalities « partially correct » or « not correct ». We then obtained the following table (Table 7):

**Table 7 :** Crossing variables « Intuitive approach to the interpretation of E(X) » and « Interpretation of particular cases of E(X) »

| Interpretation of particular cases of E(X) / Intuitive interpretation of $E(X)$ | Partially correct | Not correct | |
|---|---|---|---|
| Not relevant | 0 | 9 | 9 |
| Relevant | 1 | 6 | 7 |
| | 1 | 15 | 16 |

The intuitive approach to the interpretation of the general case of mathematical expectation is balanced between those which are relevant and those which are not (7 against 9). Now the only partially relevant interpretation of the particular cases of mathematical expectation comes from a relevant intuitive approach to the general case of mathematical expectation: this is therefore in favor of hypothesis H1: « The relevance of an intuitive approach in the interpretation of the general case of a mathematical expectation has an effect on the validity of interpretation of particular cases $E(X) = 0$, $E(X) > 0$ and $E(X) < 0$ ». The Fisher test of Ho: «There is independence between the relevance of the intuitive interpretation in the general case of mathematical expectation and the validity of interpretation of particular cases "$E(X) = 0$", "$E(X) > 0$" and "$E(X) < 0$" » against hypothesis H1| retained, leads to the probability:

$$p = \frac{C_1^0 C_{15}^9}{C_{16}^9} = 0,44$$

The test is not significant, and therefore the null hypothesis Ho cannot be rejected. Only once again, given the probability of the Fisher test $p \simeq 0,44$, we cannot here again conclude that there is strong independence between the relevance of the intuitive interpretation of $E(X)$ and that of the validity of the special cases.

### 5.2.3 *Effect of the formal approach in the interpretation of the general case of mathematical expectation on the validity of the interpretation of its particular cases*

Three out of the nineteen students interviewed (16%) adopted a formal approach in the interpretation of mathematical expectation. Here again, the formal approach, relevant or not, gave rise to an interpretation of particular cases of mathematical expectation whose validity could be correct or not correct. We therefore considered two qualitative variables F: « Formal approach to the interpretation of mathematical expectation » with two modalities « Relevant » and « Irrelevant », and I: « Interpretation of particular cases of mathematical expectation » with two modalities "correct" and "not correct". The intersection of these two variables is as follows (Table 8):

**Table 8 :** Crossing variables « Formal approach » and « Interpretation of E(X) in particular cases » among the 3 students who adopted a formal approach to their interpretation.

| Interpretation of cases of E(X) / Formal approach | Correct | Not correct | |
|---|---|---|---|
| Relevant | 0 | 1 | 1 |
| Not relevant | 0 | 2 | 2 |
| | 0 | 3 | 3 |

In view of the distribution of numbers in the previous table, we can argue that the formal approach to the interpretation of the general case of mathematical expectation has no effect on the relevance of interpretation of particular cases of mathematical expectation: proportionality in the distribution of numbers indicates a perfect independence between the formal approach of $E(X)$ in the general case and the relevance of the interpretation of "$E(X) = 0$", "$E(X) > 0$" and "$E(X) < 0$".

### 5.2.4 *Mathematical formulas for calculating mathematical expectation*

In general, mathematical formulas are mathematical modeling in a given context and under conditions that must be respected. Regardless of the validity of the mathematical formulas for calculating the mathematical expectation, they must take into account the nature of the random variable modeling a given probabilistic situation. Concerning taking into account the nature of the random variable, here are the results of the approaches provided by all of the 25 students interviewed (Table 9):

**Table 9 :** Approaches to mathematical formulas for mathematical expectation depending on the nature of the random variable

| Approaches indicating the nature of the random variable | Approaches not indicating the nature of the random variable |
|---|---|
| 48% | 52% |

The conditions of application of mathematical formulas are made to guarantee their validity while respecting the context of their applications. A priori, it is necessary to determine the nature of the random variable before giving or applying the formulas allowing the mathematical expectation to be calculated. In fact, 52% of the students surveyed introduced these formulas without mentioning the discrete or continuous nature of the random variable: one student in two does not note the importance of the nature of a random variable to express their mathematical expectation.

As for the classical properties on mathematical expectation, we obtained the following results (Table 10):

**Table 10 :** Percentage of correct formulas relating to mathematical expectation

| E(X+a) | E(kX) | E(X+Y) | E(XY) |
|---|---|---|---|
| 76% | 80% | 68% | 28% |

At the level of formulas expressing the linearity of mathematical expectation; $\mathrm{E}(\mathrm{X}+\mathrm{a})$, $\mathrm{E}(\mathrm{kX})$ and $\mathrm{E}(\mathrm{X}+\mathrm{Y})$, the majority of students interviewed gave these formulas correctly. But at the level of the formula $\mathrm{E}(\mathrm{XY})$, where there is a condition of the independence of the two random variables X et Y, only 28% of students provided this formula correctly, with the condition of independence.

Finally, note that three quarters of the students surveyed (76%) do not know that $\mathrm{E}(\mathrm{X})$ is a parameter of the random variable X. This could hinder their understanding of the centering property of a random variable and its interest, or even that of calculating a formula such as that of the covariance of two random variables. It would also be interesting to devote a whole study of educational research to questions related to mathematical expectation and the parameters of a random variable.

Indeed, more precisely, 16% of students did not provide an answer to the question about the parameters of a random variable, and 60% who provided a false answer, generally associated "parameters of 'a random variable' to the probability space on which the random variable, or its image space, or both would be defined. Here are some of the students' responses:

- Student E5: « *the parameters of the random variable are: all the events of Ω, - the sum of the probabilities equal to 1* ».
- Student E6: « *the parameters of a random variable are* $(\Omega, \mathcal{A})$, *with Ω is the universe of possibilities and A is a probability or event* ».
- Student E10: « *each random variable has parameters considered according to the experience of the random variable* ».
- Student E12: « *the parameters of the random variable are: Ω the fundamental ensemble, X the random variable, R application interval and* $\mathcal{B}_{\mathbb{R}}$ *the Borelian tribe* ».
- Student E17: « *a random variable can have parameters depending on its type. This is a Bernoulli random variable whose parameters are an integer p; a binomial random variable which takes two integers as parameters, or even a Poisson random variable which has a real* $> 0$ *, for parameter* ».
- Student E18: « *the parameters are X and Ω such as Ω is the set of all events* ».
- Student E19: « *the parameters of a random variable:* $X: \Omega \rightarrow \Omega'$ *are: random experience, fundamental space* $(\Omega)$ *and the probable space* $\Omega'$ »
- Student E22: « *type of draw (discount, without discount, simultaneously)* ».

*5.2.5 Effect of the relevance (correct or incorrect) of the definition of a probability law on the relevance (correct or incorrect) of $E(X)$*

At the very beginning of our investigation into students' conceptual representations about mathematical expectation, we questioned the students on the definition of the probability law of a random variable. These two notions being of course intimately linked, we wondered about the effect that the relevance of a definition of the law of probability can have on that of mathematical expectation. We therefore crossed the two qualitative variables L: "Relevance of a law of probabilities", by assigning it the two modalities "correct" and "incorrect", and E: "Relevance of the interpretation of mathematical expectation" with also two modalities "correct" and "incorrect". We obtained the following table (Table 11) :

**Table 11 :** Crossing variables of the variables "Relevance of a law of probability" and "relevance of the interpretation of mathematical expectation" among the 25 students interviewed

| Definition of law of probability / Interpretation of E(X) | Correct | Correct | |
|---|---|---|---|
| Correct | 6 | 5 | 11 |
| Incorrect | 2 | 12 | 14 |
| | 8 | 17 | 25 |

The crossing of this table shows that 75% of students who provide a relevant definition of the law of probability make a correct interpretation of E(X). Furthermore, 70% of students who provide an incorrect definition of the law of probability make a false interpretation of E(X). These two results are therefore in favor of hypothesis H1: "The relevance of an approach to the definition of the probability law has an effect on the relevance of the interpretation of E(X)". The Fisher test of Ho: "There is independence between the relevance of the definition of the law of probability and the relevance of the interpretation of E(X)" against the hypothesis H1| retained, gives us the following probability:

$$p = \frac{C_8^6 . C_{17}^5 + C_8^7 . C_{17}^4 + C_8^8 C_{17}^3}{C_{25}^{11}} \simeq 0{,}0433$$

The Fisher test is thus significant, the null hypothesis Ho| will be rejected: we can therefore conclude that good mastery of the definition of the law of probability of a random variable has an effect on the correct interpretation of mathematical expectation.

*5.2.6 Interpretation of mathematical expectation in a game of chance situation*

Situations of games of chance are historically at the origin of the genesis of probabilities, and first of all of the notion of mathematical expectation, even before the notion of random variable was explicitly established. Inspired by this epistemological approach, we questioned our students on the meaning of the mathematical expectation of a random variable representing the winnings of a game of chance, by asking them with justification if this could help the player to take a decision. These are interrogations on "$E(X) = 0$", "$E(X) > 0$" and "$E(X) < 0$" in a very specific context, that of a game of chance, where $X$ represents a player's winnings. Students were also asked to provide an example illustrating their answer.

Half of the students (56%) say that mathematical expectation helps to make a decision in a game situation, but none of them can give a completely relevant justification. Here are some of their responses:

- Justification from student E1: "because it specifies the probability": For student E1 the mathematical expectation is a probability.
- Justification from student E3: " $E(X)$ represents an idea of the player's chance": Here again the word "chance" refers to a probability.

- Justification from student E6: “Yes, if the value of $E(X)$ is greater, it will help him make a decision regarding this game”: E6’s justification remains partial.
- Justification from student E9: “Yes, of course because if $E(X) \geq 0$ the chance of winning is greater” : The justification of E9 is also partial.
- Justification from student E11: “Yes, if $E(X) = 0 \Rightarrow$ we have not yet made the draw, if $E(X) > 0 \Rightarrow$ the draw is made and it is positive, $E(X) < 0 \Rightarrow$ the draw is made but it is negative”: The response from E11 is inconsistent.
- Justification from student E18: “If the player wins the game, then $E(X)$ is very greater, i.e. $E(X) > 0$ ”: E18’s response is still partial.
- Justification from student E20: "Yes, if the $E(X)$ is negative, the player will have the risk of losing, if $E(X) > 0$, the player will have the chance of winning": The justification of the student E20 is almost relevant, because it does not mention the case where $E(X) = 0$.
- Justification from student E23: “Yes, if $E(X)$ is greater (or positive) there is a greater chance of winning”: The justification is also partially correct.

We therefore cross-referenced the students' responses and their justifications, distinguishing between those who provided a partially correct justification (Table 12):

**Table 12 :** Crossing of variables « $E(X)$ helps to decide in a game » and « Validity of the justification of the choice provided » among the 25 students interviewed

| Justification / E(X) helps to decide in a game | Partially correct | Not correct | |
|---|---|---|---|
| Yes | 5 | 9 | 14 |
| No | 0 | 11 | 11 |
| | 5 | 20 | 25 |

The Crossing of the variables " $E(X)$ helps to decide in a game" and "Validity of the justification", shows that all those who do not recognize the fact that the mathematical expectation of a player's gain can help the latter to making a decision during the game, all provided a false justification. On the other hand, almost one in three students (35.7%) among the students who are in favor of the expectation of winning being an aid to decision-making in a game, manage to partially justify their answer.

Thus, the distribution of students' responses and their justifications is rather in favor of hypothesis H1: “the relevance of the response depends on the validity of the justification”. Fisher's test of the null hypothesis Ho|: “students’ responses are independent of their justifications” against its opposite H1|, leads to the probability:

$$p = \frac{C_5^5 . C_{20}^9}{C_{25}^{14}} \simeq 0{,}037$$

Thus, the Fisher test is significant at the 4% threshold, we must then reject the null hypothesis Ho|, and conclude that recognizing that the mathematical expectation of winning in a game is recognized as a decision aid in such a situation, supposes a correct justification of such an interpretation. Let us note in passing that in our observations, all the students who succeeded in this question only provided a partial justification: they did not interpret exhaustively all the cases relating to the signs of the expected gain $E(X)$.

To clearly understand the conceptualization of the justifications provided regarding the expectation of gain, we asked the students to give an example illustrating their justification. Indeed, the most relevant examples provided by the students surveyed represent 36%: these examples are limited only to the citation of a game of chance, where the random variable represents a player's winnings. The intersection of the validity of the examples provided, represented by the variable E: “Example illustrating the fact that the expectation of winning helps to decide in a game”, for which we have retained the two modalities “partially correct” and “not correct », and the variable R:

"Recognition of the expectation of winning as a decision aid in a game", with the two modalities "True" and "False", gives the following table (Table 13):

**Table 13** : Crossing of variables « E(X) helps to decide in a game » and « Validity of example provided » among the 25 students interviewed

| Example / E(X) helps to decide in a game | Partially Correct | Not correct | |
|---|---|---|---|
| True | 3 | 11 | 14 |
| False | 0 | 11 | 11 |
| | 3 | 22 | 25 |

All the students who answered falsely to the fact that the expectation of winning can help decide in a game provided an irrelevant example. Furthermore, a large majority (78.5%) of those who are in favor of the fact that the expectation of gain is an aid to decision-making provided an irrelevant example. Only 21.5% of these provided an example that was partially correct. The distribution of the previous table seems to be in favor of hypothesis H1: "the relevance of the example (even partial) has an effect on the recognition of the expectation of winning as an aid to decision-making in a game". The Fisher test applied to the null hypothesis Ho, opposite of H1, which translates Ho: "independence between the relevance of the example provided to illustrate the recognition (or not) of the expectation of gain as a decision aid in a game", gives the probability:

$$p = \frac{C_3^3 . C_{22}^{11}}{C_{25}^{14}} \simeq 0{,}158$$

This test is not significant, neither at the 5% threshold nor at the 10% threshold (standard thresholds), we cannot therefore reject the null hypothesis Ho|: this can very well be explained by the fact that no student gave a concrete and relevant example.

### *5.2.7 Recognition of the notion of mathematical expectation through its mathematical formula*

In a situation of calculating the value of the mathematical expectation of a random variable whose values are 1, 2, 3 and 4. We proposed to the students the value $E(X) = \frac{1+2+3+4}{4}$ as the value of mathematical expectation. In fact, 76% made the right choice and 68% of them correctly justified their choice. Apparently, the right choice was guided by a good justification. We therefore crossed the qualitative variable C: "Choice of "True" or "False" that $E(X) = \frac{1+2+3+4}{4}$ » by assigning it two modalities, "correct" and "not correct", with the qualitative variable J: "Justification of choice", with the two modalities "correct" and "not correct" (Table 14):

**Table 14**: Crossing of variables "Choice of "True" or "False" that E(X) $= \frac{1+2+3+4}{4}$» and « Justification for choice » among the 25 students interviewed

| Justification / True or false | Correct | Not correct | |
|---|---|---|---|
| Correct | 13 | 7 | 20 |
| Not correct | 0 | 5 | 5 |
| | 13 | 12 | 25 |

The "correct" and "incorrect" justifications are balanced in the intersection of the previous table. On the other hand, all the students who answered that the proposed value of $E(X)$ represented that of a random variable taking the values 1, 2, 3 and 4, provided an incorrect justification. In addition, 65% of students who answered this question correctly justified their answer. The distribution of the previous table is therefore in favor of hypothesis H1: “the interpretation of the particular case of $E(X)$ proposed in terms of mathematical expectation is linked to the relevance of the justification provided”. We therefore applied the Fisher test to H1 and its opposite H0: “Independence between the interpretation of the particular case of $E(X)$ proposed in terms of expectation and the relevance of the justification provided”. We then obtained the probability:

$p = \frac{C_{13}^{13}.C_{12}^{7}}{C_{25}^{20}} = 0{,}015$

Thus, Fischer's exact test is very significant at the 2% level, and therefore we can reject the null hypothesis H0. We can therefore conclude that a good mastery of the mathematical expectation of a random variable requires above all a good interpretation of the terms of its formula, both in the general case and in particular cases; it of course remains closely linked to a good mastery of the law of probability of a random variable. These are important didactic elements to which we should pay close attention when teaching random variables, with a view to later gaining a good understanding of probabilistic modeling.

## 6. Conclusion

Exploring the conceptual representations of students in a probabilistic modeling situation obviously involves exploring their conceptual representations of a random variable. However, this prospecting cannot be relevant, and rightly so with reference to the historical evolution of the emergence of the definition of a random variable, without a prior didactic study on their conceptual representations of the parameters of a random variable such than mathematical expectation. Beyond what this parameter represents, the mathematical expectation, for a given random variable whose definition remains closely linked to the notion of law of probabilities of a random variable.

By questioning students on what a probability law represents, 75% of the answers are provided formally, and only 32% of the answers (which are also formal) are relevant. Otherwise, for the remaining 25% of students, the law of probability only represents a simple calculation of probability "in relation" to a random variable, or even the probability of a result resulting from a random experiment.

These difficulties in the definition and conceptual representation relating to the law of probability will certainly not be without effect on their representations of the parameters of a random variable.

In return, a majority of 76 % manages to cite at least four usual probability laws, but this success rate will drop when they are asked to identify usual probability laws when faced with elementary statements.

Only about one student in four (28%) recognizes Bernoulli's law for obtaining a given side of a balanced die: here it is quite simply of the availability of the Bernoulli model for $p \neq \frac{1}{2}$. This availability is doubled (56%) when it comes to recognizing Bernoulli’s law $\mathcal{Ber}\left(\frac{1}{2}\right)$. The effects of the types of examples provided during the teaching of Bernoulli's law are certainly not unrelated to the previous results.

On the other hand, the Binomial law remains the best recognized and available probability law among students (48%), while the Poisson and geometric laws are the least available among students, with respectively success rates of 16% and 12%.

Before analyzing possible effects between the mastery of a probability law of a random variable and that of its mathematical expectation, we noted that one in four students interprets a mathematical expectation as a probability of event, especially when it comes to giving an interpretation to "$E(X) = 0$", "$E(X) > 0$" and "$E(X) < 0$" . Although the students' approach remained predominantly formal (84% among those who gave an interpretation), it was not initially possible to measure the effect of the formal or intuitive approach on the relevance of students' interpretation of mathematical expectation in the general case. On the other hand, thanks to Fisher's exact test, it is different for particular cases"$E(X) = 0$", "$E(X) > 0$" and "$E(X) < 0$" : the formal or intuitive nature retained in their interpretation of these particular cases has an effect on the relevance of this interpretation. The intuitive approach would seem a priori to be more favorable than the formal approach, but this remains a hypothesis to be confirmed.

We therefore tested using Fisher's exact test the effect of the formal approach in the interpretation of mathematical expectation (general case) on the validity of its particular cases ("$E(X) = 0$", "$E(X) > 0$" and "$E(X) < 0$"): the conclusion is striking, there is total independence between the formal approach of $E(X)$ and a relevance of interpretation of previous particular cases.

Once again, the intuitive approach in a probabilistic situation would seem to play a didactic role in the interpretation of mathematical expectation, particularly for particular cases "$E(X) = 0$", "$E(X) > 0$" and "$E(X) < 0$" .

As for the interpretation of mathematical expectation as a parameter of a random variable, this seems to be unknown by a majority of 76% of the students questioned, while in the same proportions (between 68% and 80%) the students have a perfect command of the standard formulas for the linearity of mathematical expectation. Note also that the expectation of the product of two independent random variables is less controlled compared to the rest of the formulas on mathematical expectation, it is only 28% successful.

To the initial question relating to the effect of the relevance (correct or incorrect) of the definition of a law of probability on the relevance (correct or incorrect) of mathematical expectation, we were able to establish thanks to the test of Fisher (highly significant at the 4% level), that a good mastery of the definition of a probability law has a certain effect on a good interpretation of mathematical expectation.

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