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Analysis of Students' Reasoning in Solving Comparison Problems

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Abstract

The objective of this article is to analyze the degree of analyticity in the reasoning of both the 6th grade of elementary school and the first year of middle school, related to activities called comparison problems which are qualified as conducive to emerge analytical reasoning characterizing algebraic thinking according to several researchers (Bednarz et al., 1996; Radford, 2010; Squalli, 2000). The results we found show that a powerful algebraic potential exists in these types of activities. Moreover, they show that the letter's presence or absence did not prevent the students from deploying sophisticated analytical reasoning.

Keywords: *analytical reasoning; algebraic potential; algebraic thinking; disconnected problems; literal language*

1. Introduction

The shift from arithmetic to algebraic represents a sensitive and crucial moment for researchers in algebra didactics (Bednarz & Janvier, 1996; Carraher & Schliemann, 2007; Squalli & Bronner, 2017; Squalli et al., 2020). It builds a tough transition for middle school mathematics teachers. Most of the difficulties in algebra observed in middle school students are due to the arithmetic-algebra transition between two domains considered isolated according to old didactic approaches. However, according to the "Early Algebra" current, the development of algebraic thinking must begin in the arithmetic phase (primary) by proposing arithmetic activities with algebraic potential such as comparison problems that are considered according to several researchers in this current (Adihou et al., 2015; Saboya et al., 2014; Marchand & Bednarz, 1999; Marchand & Bednarz, 2000; Bednarz & Janvier, 1994; Bednarz & Janvier, 1996) as activities that allow students to progressively switch from the arithmetic mode to the algebraic mode by adopting analytical reasoning in the sense that leads students to proceed on unknowns (which are not necessarily represented by letters) to reach the known quantities (Squalli, 2000).

Our goal is to illuminate and document the reasoning of Moroccan students of this age group in solving algebraic problems. We assume that Moroccan students' performance in solving algebraic problems is the same before and after learning algebra in school. Would it be possible to encounter the contribution of analytical reasoning early on?

To reach this goal, we try through this article to achieve the following sub-objectives:

1. Study the impact of problems' structures (type of relationship between the unknown data) on the student's performance.
2. Discuss the degree of analyticity that characterizes the reasoning produced by students in problem-solving before and after the introduction to algebra.
3. Identify the strategies and methods mobilized by students of the 6^{th}-grade of elementary school and the first-year of middle school.

The choice to propose comparison activities is not random, but because these types of activities represent a bridge between connected problems that students are already used to solving proceeding on the data of the situation to arrive at an unknown quantity and disconnected problems that are not easily accessible by arithmetic methods except for trial-error reasoning (Adihou et al., 2015; Squalli et al., 2020).

2. Continuity and Interruption between Arithmetic and Algebraic Reasoning

Research in didactics shows that there is a double epistemological interruption between arithmetic and algebra: Algebraic reasoning is manifested by operating on the relations between the data and the unknown numbers of the problem in question and by using a formal treatment to solve it. On the contrary, arithmetical reasoning is based on the calculation of the unknown starting from the known data of the situation (Vergnaud, 1988).

Vergnaud (1988) also mentions the evolution of the objects' status and the opposition of the apprehension modes of the algebraic and numerical writings, as an example, the sign of equality used in arithmetic as a tool that announces a result. The equal sign conception leads some students to commit errors during the realization of a succession of calculations of the type 3+2=5x6=30. However, equality must be treated as a relation of equivalence because when using algebraic expressions, announcing a result will not exist since these expressions contain unknown quantities.

Kieran (1992) talks about the false continuity and discontinuity between arithmetic and algebra. These two modes share the same symbols and signs but with different meanings. For example, letters also have their significance. They are unfamiliar to students since they change. This interruption is called false continuity because it is not visible. The discontinuity is manifested through using new objects in problem solving (the letters). It can be said that during the transition from the arithmetic to the algebraic mode, a migration of symbols and signs takes place in a way that is unusual for students' conception because their meanings also change during this migration.

Arithmetic reasoning: It is marked by its arithmetic approach adopted during the resolution of problems. According to this approach, we carry out calculations on known quantities to determine what is unknown. Only the connected problems are accessible by this type of reasoning since the student has to operate on the data of the problem to find the unknown which they seek (example 1).

Example 1: Samia has 800 DHS. How much does she have left if she buys two dresses at 236 DHS each?

The unknown we are looking for here is the amount left to Samia after the purchase. The student operates on the known quantities of the situation (800; 2; 236): 800-2 x 236 by determining the operations to be performed (a subtraction and a multiplication).

Algebraic reasoning: Unlike arithmetic reasoning, it is characterized by operating on unknowns in order to arrive at known quantities. In this case, we talk about analytical reasoning in which the presence or absence of letters representing the unknowns and the equations are not decisive, but they can remain silent (Example 2).

Example 2: Maria has five times more dresses than pants. How many dresses and pants does Maria have? (Knowing that she has 24 pieces of both types of clothing in her wardrobe).

In this type of problem, the student cannot arrive at the solution by operating on the known quantities of the situation (5 and 24) except by using trial and error or the false position method. Using analytical reasoning, the student performs the operation 24:6 to find the number of pants. Indeed, the number 6 is not in the data of the problem, but it represents how many times the number of the pants is repeated if we exchange each dress by five pants. The reasoning is thus analytical even if the unknown remains silent and invisible. The arithmetic issues are not closed. Indeed, another method called false position can be useful. In this approach, we initialize the number of pants to 1. In this case, the number of dresses will be five and the total number of pieces becomes 6. So, we must multiply 6 by 4 to have 24 pieces. Then, we multiply the initialization number 1 by 4. Finally, the number of pants is 4. This method is not always relevant except in cases of proportionality. There is also the outcome of the trial and error. We give an unspecified value to the number of pants and multiply it by 5. Then, we make the sum. If we find a number higher than 24, we choose a lower value and we proceed thus until we find the exact value. The more the choices are multiple, the more this method will be less relevant and expensive.

Algebraic reasoning is characterized by a proclivity to symbolize and operate on symbols and a structural understanding of algebraic expressions (Adihou et al., 2015). According to (Bednarz & Janvier, 1996), disconnected problems represent an auspicious opportunity for entry into algebra because they allow for a smooth transition from arithmetic to the analytic mode of reasoning where students can manipulate the unknown without even representing it.

These researchers distinguish between arithmetic and algebraic problems. The first one is called "connected problems": the relationship between two known data points can be easily established, allowing students to think in an arithmetic manner. The second one is called "disconnected problems": there is no direct path between two known data points that allow for a relationship. The following figure shows an illustration of these two types of problems (Bednarz & Janvier, 1996, p. 123):

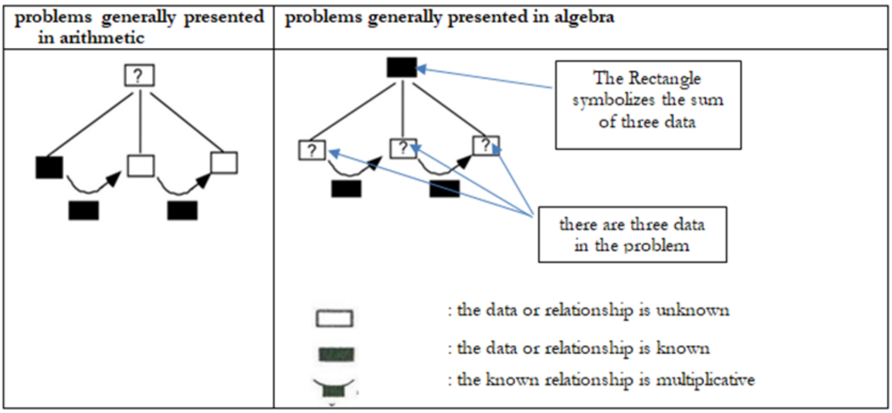


Figure 1: Arithmetic and algebraic problems (Bednarz & Janvier, 1996, p. 123)

3. Early Algebra and Algebraic Thinking

Early Algebra is a didactic stream that focuses on the early teaching-learning of algebra. It has emerged since the 2000s under the assumption that algebraic thinking could be constructed and enhanced in elementary school students, even before the introduction of formal symbolism specific to algebra. Based on the work of (Radford, 2014 ; Kieran & al, 2016), Early Algebra researchers define four indispensable elements of algebraic thinking : (1) generalization related to numerical and geometric pattern activities (patterns), (2) generalization related to properties of operations and numerical structures, (3) representation of relationships between quantities, and (4) introduction of alphanumeric notation.

On the other hand, Radford (2014) characterizes algebraic thinking by its three characteristics:

- Indeterminacy : the presence of indeterminate quantities (unknowns, variables, parameters, etc.);
- Denotation : these indeterminate quantities can be represented by symbols, gestures, or others ;
- The analyticity : i.e. that the unknowns can be manipulated as if they were known and that we can operate on them. From this perspective, algebraic reasoning is characterized by the presence of analyticity based on processing known and unknown data from properties. However, for arithmetic reasoning, this characteristic is not satisfying as in the case of trial-and-error reasoning (Radford, 2014). Therefore, Analytical reasoning remains an indicator of the development of algebraic thinking in the context of problem-solving. It consists of considering the unknowns, representing them by letters, operating on these symbols to form relations and equations, and finally, finding the values of the unknowns (Squalli et al., 2020).

4. The Analysis Model

The analysis model we propose is essentially based on the model proposed by Squalli et al. (2020).

4.1 Degree of analytical reasoning

Three broad categories of reasoning are considered (Squalli et al., 2020) : The first category includes non-analytic reasoning. The second category includes analytical reasoning. The third category includes reasonings called analytical tendencies that have a degree of analyticity that is not optimal.

4.1.1 Reasonings of a non-analytical nature

These are purely arithmetic reasonings in which the student operates on the known data and relationships to find the unknowns. This kind of reasoning is appropriate for solving connected problems.

4.1.2 Analytical reasoning

These are reasonings that meet the criteria of analytic reasoning : (1) consideration of the unknown, (2) its representation by a symbol or letter, (3) expression of the relationships between the known data and the unknowns in the problem based on the denotation of the unknowns, and (4) operating on these representations to simplify the equation and find the values of the unknowns.

4.1.3 Reasoning with an analytical tendency

This category includes three different types of reasoning. The first is hypothetico-deductive reasoning, in which the student assigns a value to an unknown quantity knowing it to be false. Then, operates on the relations and generates the values of other unknowns. After that, the student uses the relationships and the generated values to determine the exact value of the original unknown. False position reasoning is one example that can illustrate reasoning with an analytical tendency. The student acts as if the value of the unknown were known, but instead of operating on a representation of the unknown, he or she operates on a false but determined value (see example on page 3). The second type includes reasoning in which the student considers the unknowns as variables for a brief time. To find the values of these variables that satisfy the conditions of the problem, he does not operate on them in analytical reasoning, but on their numerical instantiations. This is an example of functional reasoning. The third type of reasoning includes reasoning in which the student considers the unknown, assigns it a representation, and uses this representation to translate the relations between the unknown and the known, but does not rely on these representations to determine the values of the unknown. This makes the level of analyticity low and perhaps considered non-optimal.

4.2 Category of reasoning and nature of the register of semiotic representation

The registers of semiotic representations represent a means of the appearance of mental representations of an abstract idea. They are the results revealed by the signs derived from a system of representation (figure, statement in everyday language, algebraic formula or expression, graph, diagram, etc.), (Duval,1991).

- Numerical register: only specific numbers and operations on these numbers are included in the traces of the student's resolution.
- Algebraic register: the student uses conventional algebraic terminology. He/she notes the indeterminate with a symbol or a letter detached from the context.
- Intermediate register: the student uses non-algebraic and non-numerical representations.

Table 1 summarizes the categories of reasoning according to the degree of analyticity and the nature of the register of semiotic representation :

Table 1: Categories of reasoning according to the degree of analycity and the nature of the register of semiotic representation (Squalli et al., 2020).

arithmetic reasoning	Reasoning with analytical tendency	Analytical reasoning
•Direct calculation; numerical register • Trial and error, simple adjustment; numerical register •Trial-and-error, Reasoned adjustment	•Functional reasoning, intermediate register (table of values) •False position type, numerical register, intermediate register •Explicit unknowns, conventional algebraic register but without operation on the relations between the unknowns.	•Unknowns are not presented explicitly, and the register is numeric •Use an intermediate Unknown, and the register is numerical •The register used is conventional algebraic, but it remains linked to the context •The register used is conventional algebraic and has no link with the context

5. Research Methodology

In order to document the reasoning of Moroccan students in solving algebraic problems, we confront the students individually with a test of three problems and without the intervention of the teacher explaining that the study will be conducted voluntarily.

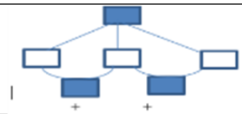
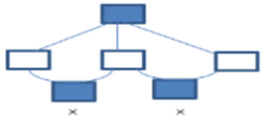
5.1 Study Group

The study group contains a total of 411 students from 9 schools. 204 of them are first-year middle school students (aged 12-13) from 5 schools (private and public), and 207 are 6th -year primary students (aged 11-12) from 4 schools (private and public). The 6th -grade students have never taken an algebra course, while the first-year students have already been introduced to algebra through a course entitled "equations", followed by chapters on the development and the factoring of algebraic expressions.

5.2. Choice of problems

The problem statements with comparison relationships are presented in Table 2:

Table 2: Statements of compositional structure problems with their comparison relations

Problem	Structure and Nature of Relationships
Problem1: Mounir, Ahmed, and Aya together have 80 books. Mounir has 15 more books than Ahmed, and Aya has 20 more books than Mounir. How many books does each have?	
Problem2: In an electronic game, the three brothers, Amine, Reda, and Sami, together scored 136 points. Amine scored 12 points more than Reda, and Sami scored twice as many points as Amine. What is the score of each of the three brothers?	
Problem3: Mounir, Ahmed, and Aya have 420 books together. Can they share them as follows? Ahmed takes double the number of books that Mounir has, and Aya takes triple the number of books that Ahmed has.	

The three problems are disconnected problems (Bednarz and Janvier, 1996) of compositional structure : one of the data is the end point of a relation will be the starting point of the other relation, which are distinguished according to :

- The relations with the data (additive, multiplicative, or additive and multiplicative)
- In the third problem, we invited students to verify if the proposed way of sharing is possible or not and to reveal the methods and approaches that they will follow to demonstrate the impossibility of this sharing. The students used to find the share of each of the partners in the previous problems. On the other hand, in this one, they have to arrive at a contradiction that leads them to the correct answer. The chosen quantities are relevant didactic variables that can influence the approach of the resolution adopted by students. They are invited to verify if the proposed way of sharing the books is possible or not. The data of the problem are chosen so that the smallest share is a rational number. The objective is to verify if the student attributes a meaning to the obtained result and to associate it with the results. That is to say his capacity to verify the results found. Indeed, the proposed sharing is impossible since the number of books is not divisible by 9, knowing that the number of books can only be an integer.

5.2 *Data analysis*

In addition to the descriptive analysis that allows us to give a general view of the students' resolutions to each of the problems, we also used qualitative analysis to identify the procedures mobilized in terms of the analytics adopted in the resolution of inequitable sharing type problems.

6. Results

6.1 Distribution of productions analyzed concerning non-responses

Table 3: distribution of non-responses by level and problem

Problem	% by problem	6th -grade primary	1st-year middle school
Problem 1	11 ,04% (53)	69,81% (37)	30,19% (16)
Problem 2	36,25% (174)	51,15% (89)	48,85% (85)
Problem 3	52 ,71% (253)	48,62% (123)	51,38% (130)
% (NR) by grade	39% (480)	51 ,88 % (249)	48 ,12 % (231)

The above results show that there is a significant percentage of non-answers. In fact, among the 1233 productions that were analyzed, (480) 39% do not express any answer to at least one of the three problems. Not knowing the disconnected types of problems affects the increasing percentage of non-answers (Abouhanifa et al.,2018). This hypothesis is supported by the fact that the school level is not a determining variable concerning the non-answers since the total number of these last ones are shared almost equally by the two levels. 51,88% (249) present the 6th-year primary students' production, and 48,12% (231) present 1st-year middle shcooll students. However, the difference between the two school levels is representative concerning problem 1. 69,81% (37) of non-responses for the 6th-year primary and only 30.19% (16) for the 1st-year middle school.

We also notice that the non-responses rate increases from problem 1 to problem 3. The latter represents a higher rate of 52,71% (253). we note that the students are easily engaged in problems having only additive relations contrary to those with a multiplicative base regardless of the method used.

6.2 Distribution of productions analyzed concerning successful responses

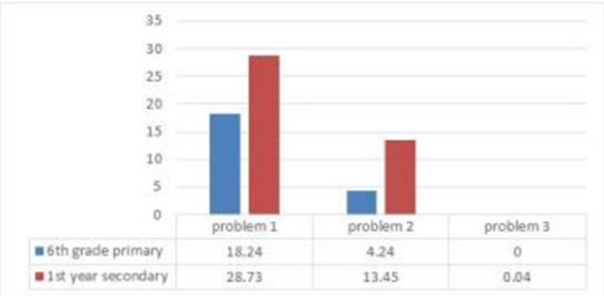


Figure 2: Distribution of productions analyzed concerning successful responses

We observe that problem 1 is the easiest to solve for both levels, with a success rate of 18.24% (31) for the 6th-year primary and 28.73% (54) for the 1st-year middle school.

The relations' nature plays an important role in a students' success. In fact, we notice that the problems with sequence composition and additive relations are more successful. This contradicts the results found by Oliveira et al. (2017), who state that whatever type of sequence of the problems (puits, composition, source), the students do well in the problems that contain two relations of the same nature (additive, additive) (multiplicative, multiplicative) which is not the case in our study.

On the other hand, we notice that the success rate of the third problem is 0% for the 6th-year of primary school and very low for the 1st-year of middle school with a percentage of 0.04%. Not because the students are not well engaged in the solution steps but, from our point of view, because the problems' nature and the type of relations (multiplicative) that hinder the steps followed by students of both levels, particularly the primary school students. The didactic variables chosen entail that the solution found is a rational number. Students, in the last stage of the resolution, do not know how to attribute meaning to the results obtained and associate it with the context. In fact, they attribute a rational quantity to the number of books, which does not make sense. Consequently, the proposed sharing is not possible. It is a problem that puts the student in a position of responsibility since the answer ends with a decision to be made (one must decide if the sharing is possible or not!).

Consequently, the numerical data of the problem and the instructions affect the success of the students' answers. This observation shows the importance of diversifying the didactic variables to offer situations that encourage work on different types of numbers.

6.3 *Categories of reasoning and distribution of these categories by grade level:*

The table below represents the distribution of different categories of reasoning for 6th-grade and 1st year secondary for each of the three problems:

Table 4: Distribution of reasoning categories in the 6th-grade primary and 1st-years middle school

Category of reasoning	Type of reasoning	6th-grade primary (372)		1st -year middle school (381)	
Non-analytical	Direct calculation	Problem 1	28 ,23% (105)	Problem 1	6 ,56% (25)
		Problem 2	20 ,16% (75)	Problem 2	8 ,92% (34)
		Problem 3	14 ,78 % (55)	Problem 3	8 ,14% (31)
	% (NR) by grade	63,2 % (235)		25,62(90)	
	Trial and error without adjustment	Problem 1	0% (0)	Problem 1	0% (0)
		Problem 2	0% (0)	Problem 2	0% (0)
		Problem 3	0% (0)	Problem 3	0% (0)
	% (NR) by grade	0% (0)		0% (0)	
	Trial and error with adjustment	Problem 1	0% (0)	Problem 1	0% (0)
		Problem 2	0% (0)	Problem 2	0% (0)
		Problem 3	0% (0)	Problem 3	0% (0)
	% (NR) by grade	0% (0)		0% (0)	
	% (NR) by grade	63,2 % (235)		25,62(90)	
With analytical tendency	Type of false position	Problem 1	0,27% (1)	Problem 1	0% (0)
		Problem 2	0% (0)	Problem 2	0% (0)
		Problem 3	0% (0)	Problem 3	0% (0)
	% (NR) by grade	0,27% (1)		0% (0)	
	Functional reasoning	Problem 1	0% (0)	Problem 1	0% (0)
		Problem 2	0% (0)	Problem 2	0% (0)
		Problem 3	0% (0)	Problem 3	0% (0)
	% (NR) by grade	0% (0)		0% (0)	
	Unknown relations and explicit equation, without operations on these representations	Problem 1	0 ,27% (1)	Problem 1	5 ,78% (22)
		Problem 2	0% (0)	Problem 2	1 ,05% (4)
		Problem 3	0 % (0)	Problem 3	0 ,26(1)
	% (NR) by grade	0 ,27% (1)		7 ,09% (27)	
	% (NR) R.T.A by grade	0 ,54% (2)		7 ,09% (27)	
Analytical	Silent Unknown or silent equations	Problem 1	2 ,15 % (8)	Problem 1	3 ,41% (13)
		Problem 2	1 ,08% (4)	Problem 2	0,52% (2)
		Problem 3	0 ,54 (2)	Problem 3	0,52% (2)
	% (NR) by grade			4,46% (17)	
	Unknowns and explicit equations without loss of context	Problem 1	0 % (0)	Problem 1	0% (0)
		Problem 2	0 % (0)	Problem 2	0% (0)
		Problem 3	0 % (0)	Problem 3	0% (0)
	% (NR) by grade	0% (0)		0% (0)	
	Unknowns and explicit equations with loss of context	Problem 1	0 % (0)	Problem 1	21 ,26% (81)
		Problem 2	0 % (0)	Problem 2	11 ,81(45)
		Problem 3	0 % (0)	Problem 3	5,77% (22)
	% (NR) by grade	0 % (0)		38 ,85 % (148)	
	% (NR) by grade	3 ,76% (14)		43 ,31% (165)	
Not identified		Problem 1	14 ,78% (55)	Problem 1	12,34% (47)
		Problem 2	10,06 % (38)	Problem 2	8 ,92% (34)
		Problem 3	7 ,53% (28)	Problem 3	4 ,72% (18)
	% (NR) by grade	32,53% (121)		25 ,98% (99)	

6.3.1 Non-analytical reasoning

The results show that non-analytical reasoning is spread among students in the 6th-grade with a percentage of 63.2% (235), while it represents only 25.62% (90) of the first-year middle school students. Direct calculation represents the totality of this non-analytical reasoning.

6.3.1.1 Direct calculation

The direct calculation is the most used by the 6th -grade elementary students in each problem. Students tend to use this procedure more when the problem contains more additive relationships: 28 .26% (105) for problem 1, 20 .16% (75) for problem 2, and 14 .78% (55) for problem 3. It appears that the presence of additive structure in the problem influences the use of this procedure by students who have not yet taken an algebra course. This procedure often leads students to a wrong answer. During this reasoning, students often use the numerical register.

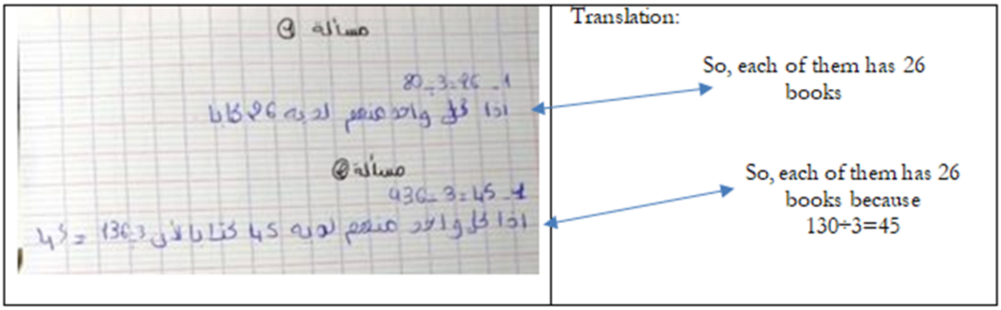


Figure 3: illustration of a response based on direct calculation, with free translation (student 1)

In figure 3, student 1 (6th-grade primary) proceeds in solving problem 1 as if the problem was of a connected type. He divides the total of books 80 by the number of subjects 3. (80 and 3 are data of the problem). But, generally, this approach leads to wrong results when the problem is disconnected. The ubiquity of this type of answers shows that students used to solve connected problems. On the other hand, the student keeps only the integer part of the result to prove the contradiction (80 ÷ 3) by performing the following operation (80 ÷ 3=26). The equality relation is incorrect because the second member of the equality must be a rational number. However, we could say that the student has a good logical interpretation of the results to be obtained which must be natural numbers as long as the quantity to be sought is the number of books, which pushes the student to keep only the integer part of the number 26, 666...

To solve problem 2, student 1 follows the same approach, which shows that the student remains confined to the approaches specified for connected problems.

In direct calculation, we notice that the nature of the additive-additive relations (problem 1) can favor a correct answer, as shown in figure 4 in the case of student 2 (6th - year primary), contrary to the additive-multiplicative and multiplicative-multiplicative relations.

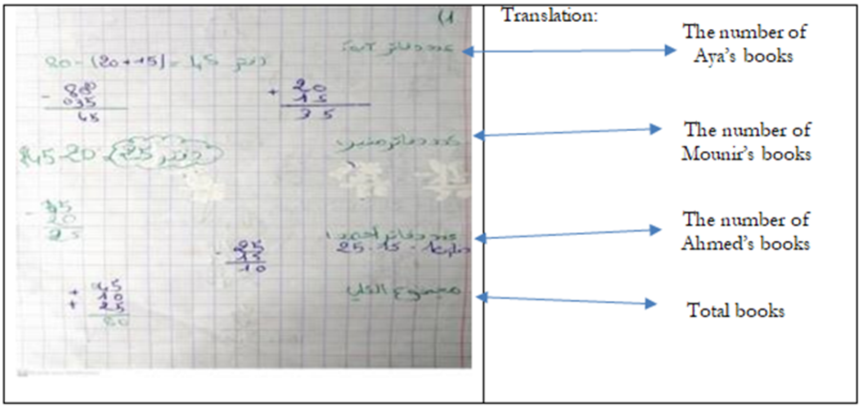


Figure 4: illustration of a correct answer based on direct calculation, with free translation (student 2)

Student 2 starts by subtracting the two additions 15 and 20 from the total of 80 to find the share of Aya. It is worth 45. Then, he subtracts 20 from the share of Aya to find the share of Mounir, which represents 25. After that, he subtracts 15 from 25 to find the share of Ahmed, which is 10. The reasoning followed by the student is incorrect even if the operations carried out and the result found are correct. This can be explained by the didactic variables' nature adopted in the problem, which are the quantities chosen. The result of this procedure would have been wrong if we had chosen other variables instead of (15,20,80). This unexpected observation led us to wonder about the conditions to be set on the data of the situation to overcome this didactic handicap.

Schematizating disconnected type problems proposed in our research (See Figure 1) assumes that there is no direct bridge between the total (known) and the other unknown quantities (Bednarz & Janvier, 1996; Saboya et al., 2014).

Following the answer of student 2 for problem 1, we see that the student was able to correctly determine the answer by operating only on the known quantities. This leads us to look for the condition that should be satisfied by problem 1, which is the additive-additive type. We find that the didactic variables should satisfy the following condition:

$$Total \neq 2 \times Addition\ relationship1 + \frac{5}{2} \times\ Addition\ relationship2$$

This condition leads us to avoid a correct result when using an arithmetic approach, and subsequently, we propose our model as follows:

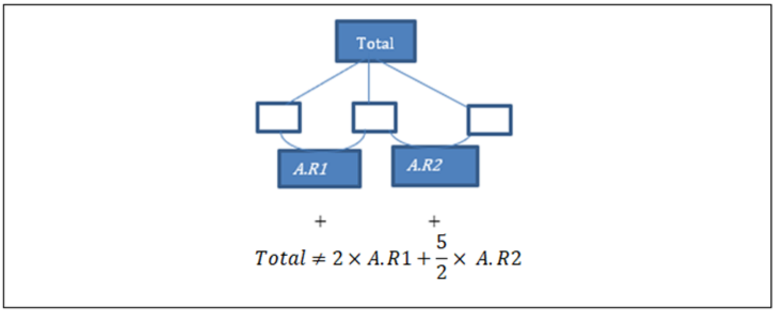


Figure 5: our proposed model for disconnected problems in the case of compositional structure and aditive-aditive relationship

Receiving an algebra course did not prevent middle school students from using the direct calculation approach, but with lower degrees compared to primary students: 6.56% (25) for problem 1, 8 .92% (34) for problem 2; 14% (31) for problem 3. we do not identify any correct answers for this category when using this type of approach.

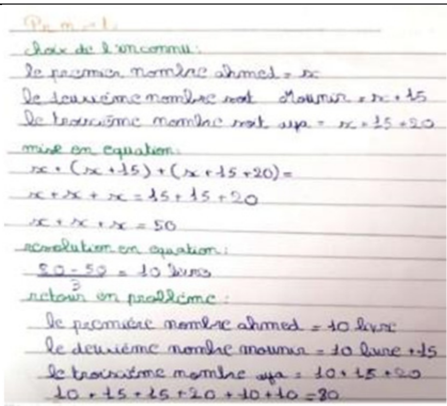
6.3.2 Reasoning with analytical tendency

Concerning the percentage of reasoning with analytical tendency, the table shows 7.09% (27) for 1st-year students and 0.54% (2) for the 6^{th}-grade.

The majority of reasonings of this type are of the nature of explicit unknowns, relations, and equations without operations on these representations. In which the pupil makes the unknowns explicit by letters and represents the relations while treating arithmetically. We note that only one production of the false position method appeared among the pupils of the 6^{th} -primary.

6.3.2.1 Explicit unknowns, relations, and equations without operations on these representations

In the category of reasoning with an analytical tendency, the 1^{st}-year students used only this type of reasoning 7.09% (27), in which the student makes the unknowns explicit by letters and represents the relations while treating arithmetically. We notice that only one production of this type was noted among 6^{th} - grade students. The example of the reasoning of student 3 (1^{st}-year of middle school). (see figure 6).



Translation:
Choice of the unknown
The first number is Ahmed =x
The second number is mounir=x+15
The third number is Aya=x+15+20
Putting in equation
x+(x+15)+(x+15+20)=
x+x+x=15+12+20
x+x+x=50
Solution in equation $\frac{80-50}{3}$ = 10 books
Back to the problem
The first number is Ahmed =10 books
The second number is Mounir =10 books+15
The third number Aya=10+15+20
10+15+15+20+10+10=80

Figure 6: illustration of an answer based on explicit unknowns, relations, and equations without operations on these representations, with free translation (student 3)

The student 3 uses the algebraic register to present the relations in a correct way. He explains the unknown "number of Ahmed's books" by x. He defines the two other unknown quantities as "number of Mounir's books" by x+15 and "number of Aya's books" by x+15+20. Then, he tries to put these relations in an equation always using this register. In this case, the student could not form the equation in the correct way. At this stage, the student is between a costly method and a bad manipulation of the rules of treatment. He decides to abandon the algebraic resolution in favor of the numerical register to have a solution as if the use of the letter is only used to give the equation, and that it is the numerical phase that follows. He names this stage by the resolution of the equation. In his opinion, these are two distinct stages. The student has not been able to escape from the numerical mode, even if he has already received a formal algebra course which raises questions about the way the unknown was introduced in middle school.

We note that this procedure is mostly used in problem-solving 1 with a percentage of 5 .78% (22)

6.3.2.2 False position reasoning, numerical register

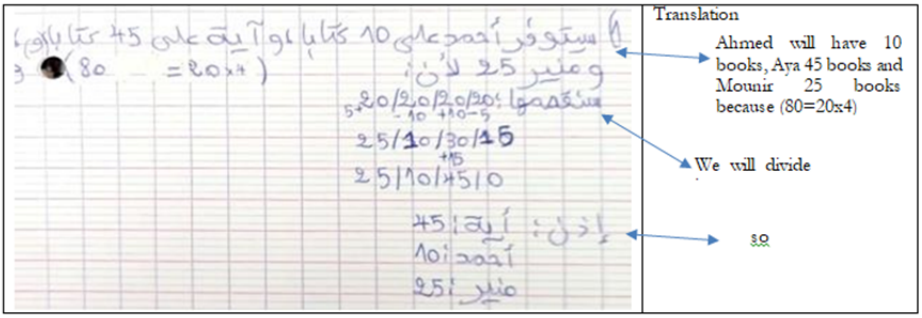


Figure 7: illustration of a response based on the false position, with free translation (student 4)

The student starts with an initial state by assigning values to the unknowns that he knows to be false, but his goal is to make the invariants of the problem appear in a numerical register. He divides 80 into four equal parts (20 × 4). He generates these values by subtracting 5 and 10 from two small parts and adding 10 to each of the other two parts (the larger ones). Then, he continues to adjust these values based on the relationships between the unknowns. He finally gets the value of the three unknowns (25/10/45).

We notice that the unknown and the equation remain implicit in this procedure, and the student masters it well by mathematizing the problem since he knows the relationship between the three unknowns and manages to determine the correct values for the unknowns of the situation.

6.3.3 Analytical reasoning

The analytical reasoning percentage represents 43.31% (165) for the first-year middle school students and 3.76% (14) for the 6^{th}-year primary students. It represents a negligible percentage.

Most of these reasonings are of the type unknowns and explicit equation with no connection with the context, and a minority of reasoning of the type unknown not explicitly represented, numerical register.

6.3.3.1 Unknown not explicitly represented, numerical register:

In this type of reasoning, the unknown and the equation are not explicit, although they are the object of the students' thoughts.

The table shows that the percentage of the students who used this procedure is 4,46% (17) for 1st-year students and 3,76% (14) for the 6th-grade students.

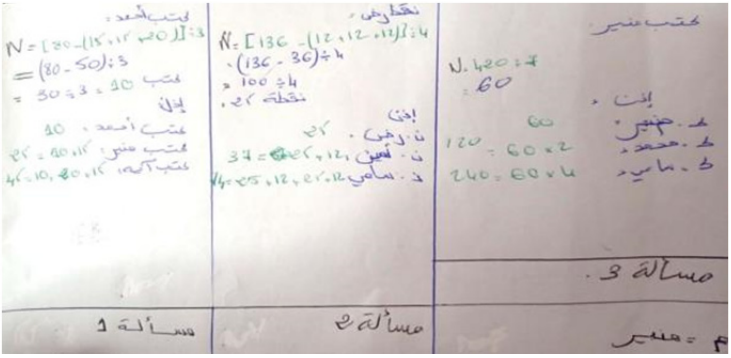


Figure 8: Illustration of an answer based on analytical reasoning where the unknown is silent (student 5)

To solve problem 1, student 5 subtracts the value (15+15+20) from the total of 80. Then, he divides the result by 3,[80-(15+15+20): 3)] to finally find the smallest share, which is Ahmed's. Of course, he did not designate the unknown by a letter or a symbol, but he proceeded with it. This student uses the same method to solve problem 2. He understands well the equation of the problem based on the subtraction of (12+12+12) from the total 136 and the division of the result found by 4. This shows that he has a good interpretation of the relations; linking the three unknowns in the two problems, which is not the case for the third problem in which the student divides by 7 instead of 9, which reflects a misinterpretation of the relations between the three unknowns. This shows that students find it more difficult to solve problems with a multiplicative-multiplicative relationship in comparison with other types.

6.3.3.2 Unknowns and explicit equation by detaching from the context:

This type of reasoning appears only in a part of first-year students who are already initiated to the algebraic notation. This part represents a global percentage of 38,85 % (148); 21,26% (81) for problem 1; 11,81(45) for problem 2, and 5,77% (22) for problem 3), while it is absent for 6th-year primary students.

Concerning this approach, we notice a decreasing success rate progressively from problem 1 to problem 3. This remark generates a variable success rate according to the relationship's structure of the problems (37.04% for problem 1, 31.11% for problem 2, and 14.29 for problem 3).

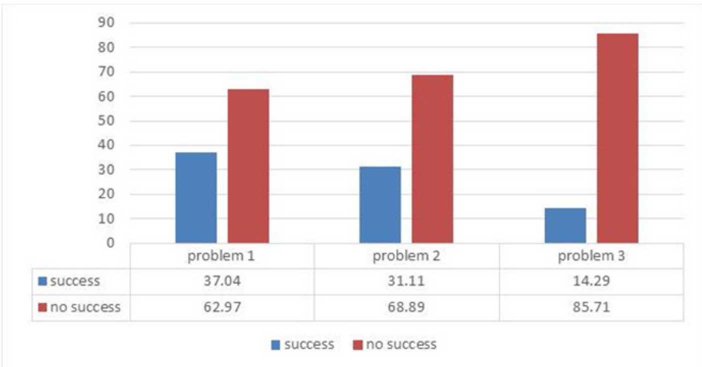


Figure 9: Distribution of 1st-year middle school students' reasoning "Unknowns, equations, explicit while detaching from the context" according to the success criteria

These students do not seem to be able to use the algebraic procedure when solving problems of different structures. We also note that only three productions are successful in problem 3 among the answers of 22 students who used this procedure, as illustrated in the following figure:

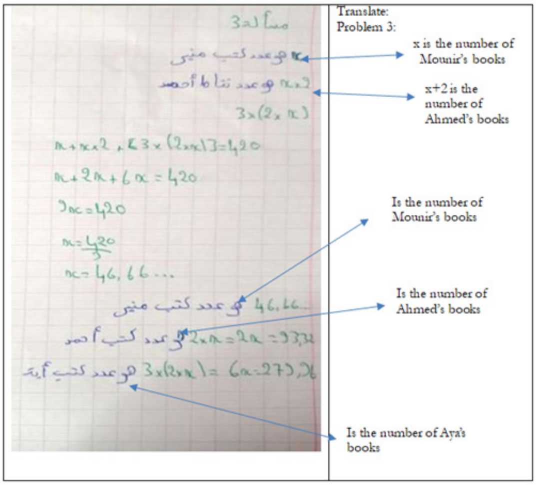


Figure 10: Illustration of an answer based on an algebraic method, with free translation (student 6)

Student 6 explained the relationships, formed an equation, simplified it, and isolated the unknown x by adopting algebraic transformations detaching immediately from the context. However, the student's answer is incomplete since he kept the rational number while the number of books is a quantity that accepts only integers as a solution. It was necessary to deduce that the proposed division is not possible.

Student 7, whose answer is presented in the figure below, understood well that the solution found cannot express the number of books, which is translated as "impossible because it is a decimal number". He understands that the proposed division is not possible.

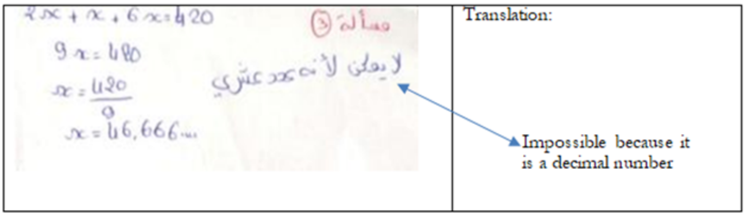


Figure11: the answer of student 7, with free translation

6.4 Unidentified methods

In this category, the student gives a result of a problem without proceeding to any identifiable approach. A significant percentage of students of both levels are in this case 25 .98% (99) for the 1st-year middle school (problem 1: 12.34% (47), problem 2: 8.92% (34), problem 3 :4 .72% (18)). On the other hand, 32.1% (121) for the 6th year primary. The percentages vary according to the problems: (14.78% (55) for problem1, 10.06%(38) for problem2, and 7,26% (28) for problem 3).

The graphs below show the different responses among two subcategories: correct and incorrect. For each of the levels:

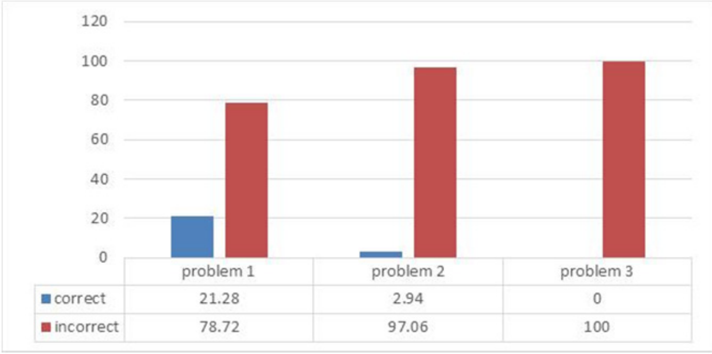


Figure 12: Percentage distribution of unidentified methods for 1st-year middle school students

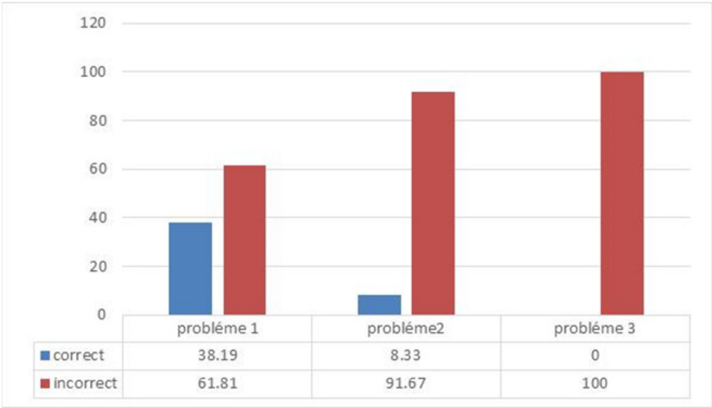


Figure 13: Percentage distribution of unidentified methods for 6th-grade students

We observe in problem3 that all the answers are incorrect, and the success increases in problem1: 38.18% for the pupils of the 6th year of primary school and 21.28% for the pupils of the 1st -year of middles chool while this percentage decreases in problem 2: 8.33% for the 6th year students and 2.94% for the pupils of the 1st-year of middle school.

According to these results, we can raise the following question: Did these students use a mental calculation or a draft that remains out of sight of the observers?

7. Discussion and Conclusion

The objectives of this research were to document the students' reasonings in order to show the importance of the activities of inequitable sharing in the development of algebraic thinking and to pay particular attention to the students' productions in terms of mobilized procedures and registers of representations. For that, we approached in our analysis three principal aspects: success rate, mobilized strategies, and the register of representation adopted.

The results show that the success rate of the problems is low among all students. It increases among secondary school students compared to those in primary school. According to our point of view, this success rate would have increased if we had submitted very early activities valuing analytical reasoning.

Another factor influencing success rates is the nature of the relationships. This has an impact on both the success rate and the type of procedure preferred by students.

The nature of the relationships was another determining factor that influenced the success rate as well as the type of procedure deployed by students of both levels.

The surprising response based on the numerical method, which led student 2 to the correct answer even though the method was incorrect, prompted us to discuss the impact of the data values of the situation and to look for the conditions to be imposed to overcome this didactic anomaly that can hinder any analytical method and consequently destroy the algebraic potential of the activity.

Our results show that most of the pupils of the 6th -year of primary school produced non-analytical

reasoning despite the presence of some productions of an analytical nature. On the other hand, we identified analytical reasoning in the pupils of the first-year of secondary school, but they were expressed in a pure numerical register as if they were not yet initiated to formal algebra.

Some first-year college students used only the letter to represent the equation without processing. They quickly abandon the algebraic register and continue in a pure arithmetic register, which indicates the limits of the arithmetic-algebraic transition (Adihou, 2020).

This phenomenon can be explained by two hypotheses:

a) Students do not master the techniques of reducing algebraic expressions and solving equations,
b) The introduction of the letter as a problem-solving tool was not introduced in a correct way that presents it as a number whose value is not known.

The verification of these two hypotheses can be the object of a future work that is based on the study of the introductory activities of algebra in secondary school. Moreover, analyzing the activities of literal calculation proposed in 1st-year secondary classes during the phase of transition arithmetic-algebra.

The use of the arithmetic method by secondary school students can also be explained by the fact that they are used to connected problems that represent a zero degree of analyticity during the primary school years. In this sense, the didactic choices of the primary school teachers can be an obstacle to the use of analytical reasoning.

We notice that even though some first-year students were introduced to algebra, they have produced reasoning with an analytical tendency in a numerical register. If they had been introduced to disconnected problems, they would have had the time to develop their reasoning.

The explanation would be that middle school students have already confronted disconnected problems, although they have not yet mastered them in a way that will allow them to complete the treatment in the algebraic register.

Research shows that most first-year middle school students have used analytical reasoning of the unknowns and explicit equations type with no connection to context, which shows that the introduction to the conventional algebraic method creates obstacles to partitioning other types of reasoning (Bednarz & Janvier, 1994).

In contrast to problems of an additive-additive nature, we note students' low engagement in multiplicative-multiplicative problems.

On the contrary, Oliveira and Rhéaume (2014) state that students tend to proceed with the analytical procedure where the unknowns are not presented explicitly in the numerical register when it comes to a problem of a multiplicative-multiplicative nature. Therefore, it is recommended that teachers use these types of problems to encourage their students' analytical reasoning.

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